5.1. Finding Primal Basic Feasible Solutions

We noted in Chapter 3 that the primal feasible region $\mathcal{F}_P$ has a finite number of extreme points. Since each such point has associated with it a basic feasible solution (unique or otherwise), it follows that there exists a finite number of basic feasible solutions. Hence an optimal solution to the primal linear programming problem will be contained within the set of basic feasible solutions to $AX = b$. How many elements does this set possess? Since a basic feasible solution has at most $m$ of $n$ variables different from zero, an upper bound to the number of basic feasible solutions is

$$\left(\begin{array}{c} n \\ m \end{array}\right) = \frac{n!}{m!(n-m)!},$$

i.e., we are interested in the total number of ways in which $m$ basic variables can be selected (without regard to their order within the vector of primal basic variables $X_B$) from a group of $n$ variables. Clearly for large $n$ and $m$ it becomes an exceedingly tedious task to examine each and every basic feasible solution. What is needed is a computational scheme which examines, in a selective fashion, only some small subset of the set of basic feasible solutions. Such a scheme is the simplex method (Dantzig (1951)). Starting from an initial basic feasible solution this technique systematically proceeds to alternative basic feasible solutions and, in a finite number of steps or iterations, ultimately arrives at an optimal
basic feasible solution. The path taken to the optimum is one for which the value of the objective function at any extreme point is at least as great as at an adjacent extreme point (two extreme points are said to be adjacent if they are joined by an edge of a convex polyhedron). For instance, if in Figure 5.1 the extreme point $A$ represents our initial basic feasible solution, the first iteration slides $f$ upwards parallel to itself over $X_P$ until it passes through its adjacent extreme point $B$. The next iteration advances $f$ in a similar fashion to its optimal (maximal) basic feasible solution at $C$. So with $f$ nondecreasing between successive basic feasible solutions, this search technique does not examine all basic feasible solutions but only those which yield a value of $f$ at least as large as at the preceding basic feasible solution.

To set the stage for the development of the simplex method, let us