CHAPTER 2

THE MATHEMATICS OF DISCONTINUITY

"On the plane of philosophy properly speaking, of metaphysics, catastrophe theory cannot, to be sure, supply any answer to the great problems which torment mankind. But it favors a dialectical, Heraclitean view of the universe, of a world which is the continual theatre of the battle between 'logoi,' between archetypes."

René Thom, 1975
"Catastrophe Theory: Its Present State and Future Perspectives," p. 382

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

Benoit B. Mandelbrot, 1983
The Fractal Geometry of Nature, p. 1

2.1. General Overview

Somehow it is appropriate if ironic that sharply divergent opinions exist in the mathematical House of Discontinuity with respect to the appropriate method for analyzing discontinuous phenomena. Different methods include catastrophe theory, chaos theory, fractal geometry, and synergetics theory. All have been applied in economics in one way or another.

We shall argue that what these theories have in common is more important than what divides them. They all focus on discontinuities as a fundamental reference point for describing reality. In the broadest sense discontinuity theory is bifurcation theory of which these are all subsets. Ironically then what we must consider is the bifurcation of bifurcation theory into competing schools. After examining the historical origins of this bifurcation of bifurcation theory we shall consider the possibility of a reconciliation and synthesis within the House of Discontinuity between these fractious factions.

2.2. The Founding Fathers

The conflict over continuity versus discontinuity can be traced deep into a variety of disputes among the ancient Greek philosophers. But the most explicitly mathematical was the controversy over Zeno's Paradox which sought to prove the unreality of motion through asserting the impossibility of an infinite sequence of discrete events (locations) occurring within a finite time period (Russell, 1945, pp. 804-806). It can be argued that Newton and Leibniz independently developed the infinitesimal calculus at least partly in order to resolve once and for all this rather annoying paradox.

Newton's explanation of planetary motion through the law of gravitation and
the infinitesimal calculus was one of the single most important revolutions in the history of human thought. And although there were hints of doubt in Leibniz's version, largely deriving from his recognition of the possibility of fractional derivatives (Leibniz, 1695), the Newtonian revolution represented the triumph of the view of reality as fundamentally continuous rather than discontinuous.

The high water mark of this simplistic perspective came with Laplace (1814) who presented a completely deterministic, continuous, general equilibrium view of celestial mechanics. Laplace went so far as to posit the possible existence of a demon who could know from any given set of initial conditions the position and velocity of any particle in the universe at any succeeding point in time. Needless to say quantum mechanics and general relativity have brought about the complete retirement of Laplace's demon in modern science. Ironically enough the first incarnation of the Laplacean vision in economics came with Walras' model of general equilibrium in 1874 just at the point that the first cracks in the Laplacean mathematical apparatus were about to appear.

In the late nineteenth and early twentieth centuries two lines of assault emerged upon the Laplacean-Newtonian superstructure. The first came from pure mathematics and involved the invention (or discovery) of what some described as "monstrous" functions or sets. Although these were initially viewed as curiosa many have since had practical applications through chaos theory and fractal geometry. The second line of assault came directly from unresolved issues in celestial mechanics and led to bifurcation theory.

The opening shot came in 1875 when duBois Reymond publicly reported on the discovery by Weierstrass in 1872 of a continuous but non-differentiable function (Mandelbrot, 1983, p. 4), namely

$$ W_0 (t) = (1 - W^2)^{1/2} \sum_{n=0}^{\infty} W^n \exp(2\pi ib^n t) $$

where \( b > 1 \) and \( W = b^h \) with \( 0 < h < 1 \).

This function is discontinuous in its first derivative everywhere. Lord Rayleigh (1880) examined the frequency band spectrum of blackbody radiation using a Weierstrass-like function. The lack of finite derivatives in certain bands implied infinite energy, the so-called "ultraviolet catastrophe." 1 Max Planck resolved this difficulty by inventing quantum mechanics which destroyed the deterministic Laplacean model with its view of particle motion as fundamentally stochastic.

Georg Cantor (1883) discovered the next "monster" which was itself utterly discontinuous, the famous Cantor set, also known as the Cantor dust or the Cantor discontinuum. Because he spent time in mental institutions it was tempting to label as "pathological" his discoveries, which included transfinite numbers and set theory. But this set has come to be viewed as fundamental in the development of the modern mathematics of discontinuity. The original Cantor set can be constructed by the following iterative process. Take the closed interval \([0,1]\) and divide it into thirds. Remove the open middle third. Subdivide the remaining two closed intervals into thirds and remove their respective open middle thirds. Repeat this process to infinity. What is left behind is the Cantor set or dust. This is partially illustrated in Figure 2.1.

The Cantor set exhibits various apparently paradoxical properties. On the one hand it is infinitely subdividable, on the other it is completely discontinuous. Although it contains a continuum of points it has zero length (Lebesgue measure zero). This last fact can be understood because the length of what is removed from the unit interval adds up to one. A two dimensional version of the Cantor