PROBLEMS WITH DIFFERENT TIME SCALES
AND ACOUSTICS

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1. Introduction

In applications, the initial value problem for systems of partial
differential equations which allow solutions on different time scales
typically has the form

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} P_0(\partial/\partial x)u + P_1(u,\partial/\partial x)u + \nu P_2(\partial/\partial x)u + f(x,t), \quad t \geq 0,$$

$$u(x,0) = f(x), \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$  \hspace{1cm} (1.1)

Here $u = (u^{(1)}, \ldots, u^{(n)})'$ is a real vector function with $n$ components. The operators $P_0$ and $P_1$ are first order differential operators of the form

$$P_0(\partial/\partial x) = \sum_{j=1}^{s} A_j \partial/\partial x_j, \quad A_j = A_j^* \in \mathbb{R}^{n \times n},$$

$$P_1(u, \partial/\partial x) = \sum_{j=1}^{s} B_j(u) \partial/\partial x_j, \quad B_j = B_j^* \in \mathbb{R}^{n \times n},$$

i.e., the coefficients are real $n \times n$ Hermitean matrices. The $A_j$ are constant matrices and the $B_j$ are polynomials in the components of $u$. The operator $P_2(\partial/\partial x)$ is a second order differential operator with constant real coefficients. In applications, $\nu P_2$ represents the dissipation present in the system. The parameters $\varepsilon > 0$ and $\nu > 0$ are small constants which measure the difference in time scales and the level of dissipation, respectively. (The coefficients of $P_0, P_1, P_2$ have been normalized to be of order $O(1)$).

An example of a problem with different time scales is given by
low-Mach-number flow. In a slightly simplified form it is given by

\[
\frac{du}{dt} + \text{grad } p = \nu \nabla u
\]

\[
M^2 \frac{dp}{dt} + \text{div } u = 0, \quad M \ll 1.
\]  

(1.2)

Here \( u \) denotes the velocity vector and \( p \) represents the pressure. The problem (1.2) has the form (1.1) if we symmetrize the system by introducing \( \rho_p = \bar{p} \) as a new variable. In particular

\[
P_0 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\partial \\
\partial x \\
\partial y \\
\partial z
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\partial \\
\partial x \\
\partial y \\
\partial z
\end{pmatrix}.
\]

A discussion of the system from the point of view of different time scales has recently been presented by Kreiss, Lorenz, and Naughton (1991). See also Klainerman and Majda (1982).

Typical questions for problems with different time scales are

1) Can one choose the initial data such that the fast scale is not excited?

2) What is the interaction between the fast and the slow scale?

3) Can one derive asymptotic expressions?

2. Systems with Constant Coefficients

In this section we assume that the coefficients \( B_j \) of the operator \( P_1 \) are constant matrices. In this case, we can use Fourier expansion to transform the system (1.1) and obtain

\[
\hat{u}_t(\omega, t) = |\omega| \hat{P}(\omega) \hat{u}(\omega, t) + \hat{F}(\omega, t),
\]

\[
\hat{u}(\omega, 0) = \hat{f}(\omega).
\]

(2.1)

Here \( \omega = (\omega_1, \ldots, \omega_s) \) denotes the (real) dual variable to the space variable \( x \in R^s \) and

\[
\hat{P}(\omega) = \frac{1}{\varepsilon} P_0(i\omega') + P_1(i\omega') + \nu|\omega| P_2(i\omega'),
\]

(2.2)