Lecture 8

A Priori Bounds

1 A Priori Bounds for the Riccati Equation

We want to study the equation

\[ P_{n+1} = \phi_{n+1} \left( P_n - P_n H'_n (H_n P_n H'_n + R_n)^{-1} H_n P_n \right) \phi'_{n+1} + G_{n+1} Q_n G'_{n+1}, \]

where \( P_{n_0} = \Gamma \). Recall that \( P_{n+1} \) represents the prediction error covariance at time \( n+1 \) given data to time \( n \), while \( P_n - P_n H'_n (H_n P_n H'_n + R_n)^{-1} H_n P_n \) represents the filter error covariance, the error covariance matrix of the signal process \( x \) at time \( n \) given data to time \( n \). We have and will assume the matrix \( \Phi_n \) is invertible for all \( n \). When the processes arise from sampling of continuous time diffusion processes this assumption is fulfilled. Define the mapping \( \tau_{n+1} \) as

\[ \tau_{n+1}(P_n) \triangleq \phi_{n+1} \left( P_n - P_n H'_n (H_n P_n H'_n + R_n)^{-1} H_n P_n \right) \phi'_{n+1} + G_{n+1} Q_n G'_{n+1}. \]

Define the set \( M_d(R) \) as the set of all \( d \times d \) real-entered matrices: \( M_d(R) \) is a subset of \( R^{d^2} \). Next, define a subset of \( M_d(R) \) consisting of all symmetric \( d \times d \) real-entered matrices:

\[ SM_d(R) = \{ A \in M_d(R) \mid A = A' \}. \]
On this set, define a subset $C$ containing all positive semidefinite $d \times d$ real-entered matrices:

$$C = \{ A \in SM_d(R) \mid A \text{ is p.s.d.} \}.$$ 

The set $C$ is a cone.

**Definition 8.1** $C$ is a cone iff $C + C \subseteq C$ and $\lambda C \subseteq C$ $\forall \lambda > 0$.

For every cone there is a partial ordering: if $A \in C$ and $B \in C$, then $A \geq B$ iff $A - B \in C$.

NB: The existence of a partial ordering on a set does not imply that all elements are comparable.

Remark: Recall that the $2 \times 2$ matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is p.s.d. iff $a \geq 0$, $c \geq 0$, and $ac - b^2 \geq 0$.

Next, recall the definition of the Moore-Penrose pseudo inverse. It says that if $A \in M_d(R)$, then $A^\#$ is the unique matrix that satisfies

1. $AA^\# A = A$

2. $A^\# AA^\# = A^\#$

3. $(AA^\#)' = AA^\#$

4. $(A^\# A)' = A^\# A$.

The matrix $P = AA^\#$ is a projection onto the range of $A$ and is idempotent (i.e., $P^2 = P$). For the geometric interpretation of the Moore-Penrose pseudo inverse, see [41].

**Definition 8.2 (Duffin)** If $A, B \in SM_d(R)$, then

$$(A : B) \triangleq A(A + B)^\# B.$$