CHAPTER 1

The Fine–McMillan Recursive Quantizer Model

In this chapter we shall formulate a source coding model which, we hope, will shed new light on the waveform coding techniques presently in use. Before we give the definition of the general model, however, we would like to briefly look at (using the well-known deltamodulator as an example) the type of problems and concepts that will be the main subject of this work.

An important special case, the simple quantizer, will also be discussed. An analysis of the theoretical limits in terms of our model will close this chapter.

1.1 Source, Channel, Reproduction

We assume that the double infinite sequence of random variables (r.v.) \( \{X_n\}_{-\infty}^{\infty} = \{\ldots, X_{-1}, X_0, X_1, \ldots\} = \{X_n\} \), the message-to-be-coded or source, is stationary and \( E|X_0| < +\infty \). The coded message \( \{\beta_n\}_{-\infty}^{\infty} = \{\beta_n\} \) is a sequence of \( B \) valued r.v.'s, where \( B = \{b^{(1)}, \ldots, b^{(M)}\} \) is the channel alphabet. The reproduction of \( \{X_n\} \), or decoded message, \( \{\hat{X}_n\}_{-\infty}^{\infty} = \{\hat{X}_n\} \) is also a sequence of r.v.'s. The channel is assumed to be noiseless. The fidelity of the reproduction is defined and measured by the sequence \( \{E[\psi(X_{n-d}, \hat{X}_n)]\}_{-\infty}^{\infty} \), where \( \psi: R^1 \times R^1 \to R^+ \) is the distortion function and \( d \) is the delay of the system. If the sequence \( \{X_n, \hat{X}_n\}_{-\infty}^{\infty} \) is stationary, then the single number

\[
E[\psi(X_{n-d}, \hat{X}_n)] = E[\psi(X_{-d}, \hat{X}_0)], \quad n = 0, \pm 1, \ldots
\]

qualifies the system.

Finally, we assume that there is an \( x^* \in R^1 \) such that

\[
E[\psi(X_0, x^*)] < +\infty.
\]  (1.1.1)
1.2 The Linear Deltamodulator

The linear deltamodulator (LDM) may be considered as the prototype of the family of differential predictive (DPC) techniques which almost exclusively rule the field of waveform coding. A brief look at the LDM will give us an opportunity to point out, without going into unnecessary details, those, sometimes quite restrictive, properties of the DPC techniques which have, in our view, rigidified into a kind of dogma and prompted us to formulate a more general model.

Let the channel alphabet be $B = \{1, -1\}$, i.e., let the capacity of the noiseless channel be 1 bit. The coder of the LDM, defined as

$$\beta_n = \text{sgn}(X_n - c \hat{X}_{n-1}), \quad n = 0, \pm 1, \ldots,$$

where $c$ is some constant, codes the difference of the actual input and its "prediction" into 1 bit. The decoder, defined as

$$\hat{X}_n = c \hat{X}_{n-1} + L \beta_n, \quad n = 0, \pm 1, \ldots,$$

where $L$ is also some constant, corrects the prediction according to the coded message (see Figure 1). The convention that in the first step the "previous reproduction" in the coder and the decoder are arbitrary but identical is an organic, though not explicitly stated, part of the definition of the LDM.

The DPC procedures are more general only so far as they apply more complex rules of quantization for the prediction error as well as for the prediction. They are variants of the LDM with somewhat weaker constraints on their structure. The LDM, which is a causal, Fine–McMillan RQ (see Section 1.3) with $\log_2 M = 1$ bit channel capacity, used to play an important role in the practice of data compression (see [1], [4], [7], [48], [60]).

We will now point out a few distinctive features of, and formulate some questions about the LDM.

(i) Note that the codeword sent by the coder of the LDM is always the one for which the decoder's answer is closest to the actual input. Although

![Figure 1](image-url)