In this chapter the family of halfspaces of a nonempty linear space is converted into a join system—called a factor geometry—by defining a join operation in it in a natural way. The theory of factor geometries has its roots in the problem of constructing a geometry out of the family of rays that emanate from a given point in a Euclidean space. Such a ray geometry is implicit in classical geometry and is closely related to spherical geometry. Factor geometries and join geometries share many common properties and can be studied by similar methods. Thus in a factor geometry convexity and linearity are treated in a familiar way. However a factor geometry, as an algebraic system, differs markedly from a join geometry since it contains an identity element and its elements have inverses. The development has strong—though unforced—analogy with algebraic theories of congruence relations and factor or quotient systems.

10.1 Congruence Relations Determined by Halfspaces

If \( M \) is a nonempty linear set in join geometry \( J \), the family of halfspaces of \( M \) partitions \( J \) (Theorem 8.23). We find it pays to study the relationships among points that this partition induces.

1 The prerequisites for this chapter are Chapters 6, 8, Sections 9.1–9.4 and Sections 9.13–9.17.
**Definition.** Let $M$ be a nonempty linear set in join geometry $J$. Suppose $a$ and $b$ are in the same halfspace of $M$, or equivalently $Ma = Mb$. Then we say $a$ is congruent to $b$ modulo $M$ and write $a \equiv b \pmod{M}$.

**Geometric Interpretation.** Let $M$ be a line in $JG_5$, the 3-dimensional Euclidean join geometry (Figure 10.1). Suppose $a \equiv b \pmod{M}$. Consider first the case where $a, b \not\in M$. Then $a$ and $b$ are in a proper halfspace of $M$, and so are on the same “side” of $M$. Suppose one of $a, b$ is in line $M$. Then both are, since $M$ is a (degenerate) halfspace of $M$. Thus $a \equiv b \pmod{M}$ is seen to be equivalent to: $a$ and $b$ are on the same “side” of line $M$ or are both in $M$.

![Figure 10.1](image1)

**Theorem 10.1.** The relation congruence modulo $M$ is an equivalence relation in set $J$:

- $a \equiv a \pmod{M}$,
- $a \equiv b \pmod{M}$ implies $b \equiv a \pmod{M}$,
- $a \equiv b \pmod{M}$, $b \equiv c \pmod{M}$ imply $a \equiv c \pmod{M}$.

**Proof.** Apply the definition: $x \equiv y \pmod{M}$ means $Mx = My$. \hfill \Box

The following theorem gives a simple criterion for congruence which does not involve the idea of halfspace.

**Theorem 10.2.** $a \equiv b \pmod{M}$ if and only if $aM \approx bM$.

![Figure 10.2](image2)