CONFORMAL INVARIANCE, UNITARITY AND
TWO DIMENSIONAL CRITICAL EXponents

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ABSTRACT

We show that conformal invariance and unitarity severely limit the possible values of critical exponents in two dimensional systems by finding the discrete series of unitarisable representations of the Virasoro algebra. The realization of conformal symmetry in a given system is parametrized by a real number $c$, the coefficient of the trace anomaly. For $c<1$ the only values allowed by unitarity are $c=1-6/m(m+1)$, $m=2,3,4\ldots$. For each of these values of $c$ unitarity determines a finite set of rational numbers that must contain all possible critical exponents. These finite sets account for the known critical exponents of the following two dimensional models: Ising($m=3$), tricritical Ising($m=4$), 3-state Potts($m=5$), and tricritical 3-state Potts($m=6$).

1. INTRODUCTION

One of the most intriguing features of statistical mechanical systems and of their analogs, euclidean field theories, is the existence of special critical points where the systems are scale invariant. Correlation functions of the fluctuating fields mirror this lack of scale by transforming very simply under dilations of space. The two point function $<\phi(r)\phi(0)>$, for example, is proportional to $r^{-2x}$. The number $x$ is called the scaling dimension of $\phi$ (the anomalous dimension in field theory). The scaling dimensions of the fields determine the critical exponents of the system.

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The renormalization group has provided a satisfying conceptual framework for understanding the occurrence of scale invariant points. They are fixed points of the system under renormalization group transformations. The scaling dimensions describe the linearized behavior of the renormalization group near the fixed point. Although this viewpoint is elegant and compelling, it provides no general procedure for classifying or constraining possible fixed points. Calculations are done within the context of specific families of models.

Many known examples of fixed points display a richer symmetry than simple scale invariance. They are invariant under local rescalings: transformations of space that preserve angles but change lengths differently at different points. These are the conformal transformations. The question arises whether conformal invariance can be used to constrain or construct possible fixed point theories. This possibility was suggested by Polyakov\(^1\).

Two dimensions is an especially promising place to apply notions of conformal invariance, because there the group of conformal transformations is infinite dimensional. Any analytic function mapping the complex plane to itself is conformal. Belavin, Polyakov and Zamolodchikov (BPZ)\(^2\) have shown how the rich structure of the conformal group in two dimensions can be used to analyze conformally invariant field theories.

Many two dimensional statistical mechanical systems can also be interpreted as 1+1 dimensional quantum field theories. The distinguishing feature of the quantum theories is unitarity, equivalent to the property of reflection positivity in the statistical systems. We shall see that the quantum mechanical condition of unitarity, in the presence of the large conformal transformation group, puts a powerful constraint on the allowed physical systems.

2. CONFORMAL INVARIANCE IN TWO DIMENSIONS

Conformal invariance is simplest to describe in complex coordinates \(z=x+iy\), \(\bar{z}=x-iy\). The infinitesimal conformal transformations are \(z \rightarrow z+v(z)\), \(\bar{z} \rightarrow \bar{z}+\bar{v}(\bar{z})\). A simple basis is \(v(z)=cz^{n+1}\), \(\bar{v}(\bar{z})=\bar{z}\bar{z}^{-n+1}\). When \(n=-1,0,1\) these generate the group of fractional linear