1. INTRODUCTION.

Using the vertex operator introduced in [12], we have given two Lie-theoretic interpretations and proofs of the classical Rogers–Ramanujan identities, which state

\[ \prod_{n \geq 1} (1-q^{5n-4})^{-1}(1-q^{5n-1})^{-1} = \sum_{n \geq 0} q^{n^2}/(1-q)(1-q^2)\ldots(1-q^n), \]  
\[ \prod_{n \geq 1} (1-q^{5n-3})^{-1}(1-q^{5n-2})^{-1} = \sum_{n \geq 0} q^{n(n+1)}/(1-q)(1-q^2)\ldots(1-q^n). \]

Both of these approaches are based on a pair of level 3 standard modules for the simplest affine Kac-Moody Lie algebra \(A_1^{(1)}=\mathfrak{g}(2,\mathbb{C})^\wedge\) and on the principal Heisenberg subalgebra, say \(\hat{h}\), of \(A_1^{(1)}\). Moreover, both approaches interpret the product sides of (1.1) and (1.2) as the principal characters of the vacuum spaces for \(\hat{h}\) in the standard modules. Both approaches also interpret the sum sides of (1.1) and (1.2) by means of the \(\hat{h}\)-filtration of these vacuum spaces. The second approach [15], [16] has the advantage over the first [13], [14] that the argument which interprets the \(n\)th summand on the right-hand side of (1.1) or (1.2) generalizes naturally.

The second approach is based on the new algebras ("\(\mathcal{Z}\)-algebras") introduced in the announcement [15]. The details of this work are given in [16] for the general theory and for the standard \(A_1^{(1)}\)-modules of levels 1, 2 and 3, and in [17] for the higher level standard \(A_1^{(1)}\)-modules in the principal picture. In the

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same sense that the level 3 modules are related to the Rogers–Ramanujan identities, the level 2 modules are related to a pair of Euler identities and the higher level modules are related to combinatorial identities of Gordon, Andrews and Bressoud. For all the standard \( A_1^{(1)} \)-modules, we correspondingly have constructions of \( A_1^{(1)} \) generalizing the first vertex operator construction [12] (the case of the basic, or level 1 standard, modules), although for the higher levels, we do not now have an independent proof of the corresponding combinatorial identities, which are interpreted by and used in the module construction.

The present paper is a virtually self-contained exposition of the most important cases of this work - the level 1, 2 and 3 cases - and an introduction to the papers [15]–[17]. The notation and arguments used here usually suggest the general notation and arguments. The reader should also find the earlier expositions [9], [10] useful. For further background and bibliography, we refer the reader to the five papers just mentioned.

Vertex operators and \( \mathbb{Z} \)-algebras have turned out to be closely related to work in several different directions, discussed in a number of papers in this volume. Many times during the development of these ideas, new insight has arisen from interaction between apparently distinct theories. One such case is described in Remark 8.11. We hope that the present elementary exposition will facilitate communication between people with different perspectives.

2. THE AFFINE LIE ALGEBRA \( A_1^{(1)} \) IN THE PRINCIPAL PICTURE.

Let \( g \) be the 3-dimensional simple Lie algebra \( \mathfrak{sl}(2,\mathbb{C}) \), consisting of all \( 2 \times 2 \) complex matrices of trace 0. Take the following basis:

\[
\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_{-\beta} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.
\]

The brackets are