SOME APPLICATIONS OF VERTEX OPERATORS TO KAC-MOODY ALGEBRAS

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1. INTRODUCTION

This is an account of some of my recent work [2,3] which has involved applications of vertex operators to Kac-Moody Lie theory. For V the basic $A_{1}^{(1)}$-module in the principal realization given by Lepowsky and Wilson [13], one may use vertex operators to describe the decomposition $V \otimes V = S(V) \oplus A(V)$ of $V \otimes V$ into symmetric tensors $S(V)$ and antisymmetric tensors $A(V)$. This turns out to be precisely the decomposition of $V \otimes V$ into two "strings" of level two standard $A_{1}^{(1)}$-modules which I found in [1]. This result has a remarkable application to the construction of the hyperbolic algebra $F$ with Dynkin diagram 

In [2] Frenkel and I gave a $\mathbb{Z}$-graded construction of $F$ such that the 0, 1 and -1 graded pieces (levels) were $A_{1}^{(1)}$ extended by the derivation $d$, $V$, and its contragredient module $V^*$, respectively. The higher levels were graded pieces of quotients of free Lie algebras by a graded ideal. For level 2 these were precisely determined to be $V \wedge V \approx A(V)$ modulo a single irreducible component (the top module of the antisymmetric string), and similarly for level -2 using $V^*$ in place of $V$. This gave the first precise formula for "higher level" hyperbolic root multiplicities beyond the general formula of Moody and Berman [17]. These multiplicities have a remarkable relationship with the values of the classical partition function which has led to conjectures concerning upper bounds for all hyperbolic root multiplicities [6]. Different ways of applying vertex operators to the construction of hyperbolic algebras will be discussed by others in this workshop, but, as of this writing, none has yet explained those root multiplicities for $F$ which are known precisely.

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185
In order to extend the results mentioned above to the hyperbolic algebra with Dynkin diagram \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \) I have recently studied the decomposition of \( V \otimes V \) where \( V \) is the basic \( A_2^{(2)} \)-module. The techniques of [1,2] give this decomposition into two strings of level 2 standard modules with outer multiplicities, remarkably, equal to the coefficients of the Rogers–Ramanujan identities. From numerical data sent to me by V.G. Kac, it appears that the second level of this hyperbolic algebra consists of those irreducible components of \( V \otimes V \) whose highest weights have odd principal degree \( > 1 \) relative to \( 1 \otimes 1 \). One expects this to follow as for the algebra \( F \) from the decomposition \( V \otimes V = S(V) \oplus A(V) \) and from the identification of \( A(V) \) modulo one irreducible component with the second level of the \( \mathbb{Z} \)-graded hyperbolic algebra.

The point of interest to those at this workshop is how the vertex operator techniques used to find \( V \otimes V = S(V) \oplus A(V) \) in the \( A_1^{(1)} \) case can be modified for the \( A_2^{(2)} \) case. The proof in the \( A_1^{(1)} \) case depended on the introduction of an auxiliary vertex operator on \( V \otimes V \) with components which form a Clifford algebra, which commute with the action of the principal Heisenberg subalgebra on \( V \otimes V \), and which anticommute with the action of the real root vectors on \( V \otimes V \). In the case of \( A_2^{(2)} \) the components of the analogous auxiliary vertex operator have much more complicated relations with each other and with the real root vector action on \( V \otimes V \). In fact, current joint work with J. Lepowsky shows that one is dealing here with \( \mathbb{Z} \)-algebras [15–16]. One may hope to generalize these results to all affine algebras and apply the theory of \( \mathbb{Z} \)-algebras to the decomposition of more general tensor products.

In [3] Frenkel and I were able to construct highest weight representations for all "classical" affine algebras and superalgebras. These consist of the orthogonal series \( (D_{\ell}^{(1)}, B_{\ell}^{(1)}, D_{\ell+1}^{(2)}) \), the symplectic series \( (C_{\ell}^{(1)}, B_{\ell}^{(1)}(0,\ell), C_{\ell}^{(2)}(\ell+1)) \), and the general linear series \( (A_{\ell-1}^{(1)}, A_{2\ell-1}^{(2)}, A_{2\ell}^{(2)}, A_{2\ell}^{(2)}(0,2\ell-1), A_{4}(0,2\ell)) \). The representations are given by certain "normally ordered" quadratic expressions whose linear factors generate an infinite-dimensional Clifford or Weyl algebra. This provides representations of the affine algebras on exterior or symmetric algebras of polynomials, respectively.