CHAPTER 9
The Multivariate $t$ Distribution

If $Z$ is a $\mathcal{N}(0, 1)$ variable and independent of $S$, where $v S^2$ has a chi-square distribution with $v$ degrees of freedom, then the random variable $t = Z/S$ is called a Student's $t$ variable with $v$ degrees of freedom. The distribution of $t$ can be found in elementary textbooks, and it plays a central role in statistical inference problems concerning the mean of a univariate normal distribution with unknown variance. The multivariate $t$ distribution, defined below and studied in this chapter, is a multivariate generalization of Student's $t$ distribution.

Let $R = (\rho_{ij})$ be an $n \times n$ symmetric matrix such that it is either positive definite or positive semidefinite and $\rho_{ii} = 1$ ($i = 1, \ldots, n$). Let $Z = (Z_1, \ldots, Z_n)'$ have an $\mathcal{N}(0, R)$ distribution, and let the univariate random variable $S$ be such that (i) $S$ is independent of $Z$, and (ii) $v S^2$ has a $\chi^2(v)$ distribution. Then a natural generalization of the Student's $t$ variable is

$$t = (t_1, \ldots, t_n)' = \left(\frac{Z_1}{S}, \ldots, \frac{Z_n}{S}\right)' .$$

(9.0.1)

It is clear that the distribution of $t$ involves only $R$ and $v$. Furthermore, it follows that for $v \geq 3$ the correlation coefficient between $t_i$ and $t_j$ is just $\rho_{ij}$ (see Remark 9.1.1). Thus the matrix $R$ is the correlation matrix of $t$. 

Remark 9.0.1. It should be pointed out that in addition to the random variable $t$ defined in (9.0.1), there are other multivariate $t$ variables studied in the literature for both theoretical and applied purposes. For example, another commonly used multivariate $t$ variable is the one given in the following: For $j = 1, \ldots, N$, let $X_j = (X_{1j}, \ldots, X_{nj})'$ be independent $\mathcal{N}_n(0, \Sigma)$ variables. For

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Let \( i = 1, \ldots, n \), let \( \bar{X}_i \) and \( V_i^2 \) be given by

\[
\bar{X}_i = \frac{1}{N} \sum_{j=1}^{N} X_{ij}, \quad V_i^2 = \frac{1}{N-1} \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)^2, \tag{9.0.2}
\]

then define

\[
t* = \left( \frac{\sqrt{N} \bar{X}_1}{V_1}, \ldots, \frac{\sqrt{N} \bar{X}_n}{V_n} \right)'. \tag{9.0.3}
\]

This random variable is also called a multivariate \( t \) variable in the literature, and the marginal distribution of \( \sqrt{N} \bar{X}_i/V_i \) is Student's \( t \) with \( N - 1 \) degrees of freedom (\( i = 1, \ldots, n \)). To avoid possible confusion, only a random variable of the form in (9.0.1) will be called a multivariate \( t \) variable in this chapter, and its distribution will be called a multivariate \( t \) distribution.

**Definition 9.0.1.** The \( n \)-dimensional random variable \( t \) defined in (9.0.1) is called a multivariate \( t \) variable, and its distribution is called a multivariate \( t \) distribution with parameters \( R \) and \( v \), where \( R \) is the correlation matrix and \( v \) is the number of degrees of freedom of the distribution; in symbols, \( t \sim t(R, v) \).

The multivariate \( t \) distribution has been found useful in inference problems concerning the mean vector of a multivariate normal distribution. An example of application, dealing with simultaneous comparisons of \( n \) treatments with a control (Dunnett, 1955), is given below.

**Example 9.0.1.** For \( i = 0, 1, \ldots, n \) and \( j = 1, \ldots, N \), let \( X_{ij} \) denote the \( j \)th observation from the \( i \)th population (the zeroth population denotes the control population). Under the assumption that the \( X_{ij} \)'s are independent \( \mathcal{N}(\theta_i, \sigma^2) \) variables, where the \( \theta_i \)'s and \( \sigma^2 \) are unknown, we are interested in comparing the parameters

\[
\mu_i = \theta_i - \theta_0 \quad (i = 1, \ldots, n),
\]

simultaneously based on the sample means \( \bar{X}_i \) and the sample variances \( V_i^2 \) (\( i = 0, 1, \ldots, n \)) given in (9.0.2). Denote \( v = (n + 1)(N - 1) \),

\[
Y_i = \bar{X}_i - \bar{X}_o \quad (i = 1, \ldots, n), \quad S_0^2 = \sum_{i=0}^{n} \frac{V_i^2}{n + 1}, \tag{9.0.4}
\]

and consider the \( n \)-dimensional random variable

\[
Z = \frac{\sqrt{N}}{\sqrt{2\sigma}}(Y_1 - \mu_1, \ldots, Y_n - \mu_n)',
\]

which has an \( \mathcal{N}_n(0, R) \) distribution with \( \rho_{ij} = \frac{1}{2} (i \neq j) \). Let

\[
A_1 = \{ y : y \in \mathbb{R}^n, y_i \geq Y_i - dS_0 \text{ for } i = 1, \ldots, n \}, \tag{9.0.5}
\]

\[
A_2 = \{ y : y \in \mathbb{R}^n, |Y_i - y_i| \leq dS_0 \text{ for } i = 1, \ldots, n \} \tag{9.0.6}
\]