Lecture 1
Complex Numbers I

We begin this lecture with the definition of complex numbers and then introduce basic operations—addition, subtraction, multiplication, and division of complex numbers. Next, we shall show how the complex numbers can be represented on the $xy$-plane. Finally, we shall define the modulus and conjugate of a complex number.

Throughout these lectures, the following well-known notations will be used:

- $\mathbb{N} = \{1, 2, \ldots\}$, the set of all natural numbers;
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all integers;
- $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$, the set of all rational numbers;
- $\mathbb{R}$ = the set of all real numbers.

A complex number is an expression of the form $a + ib$, where $a$ and $b \in \mathbb{R}$, and $i$ (sometimes $j$) is just a symbol.

- $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$, the set of all complex numbers.

It is clear that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

For a complex number, $z = a + ib$, $\text{Re}(z) = a$ is the real part of $z$, and $\text{Im}(z) = b$ is the imaginary part of $z$. If $a = 0$, then $z$ is said to be a purely imaginary number. Two complex numbers, $z$ and $w$ are equal; i.e., $z = w$, if and only if, $\text{Re}(z) = \text{Re}(w)$ and $\text{Im}(z) = \text{Im}(w)$. Clearly, $z = 0$ is the only number that is real as well as purely imaginary.

The following operations are defined on the complex number system:

(i). Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$.
(ii). Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$.
(iii). Multiplication: $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$.

As in real number system, $0 = 0 + 0i$ is a complex number such that $z + 0 = z$. There is obviously a unique complex number $0$ that possesses this property.

From (iii), it is clear that $i^2 = -1$, and hence, formally, $i = \sqrt{-1}$. Thus, except for zero, positive real numbers have real square roots, and negative real numbers have purely imaginary square roots.
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For complex numbers $z_1, z_2, z_3$ we have the following easily verifiable properties:

(I). Commutativity of addition: $z_1 + z_2 = z_2 + z_1$.

(II). Commutativity of multiplication: $z_1 z_2 = z_2 z_1$.

(III). Associativity of addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(IV). Associativity of multiplication: $z_1 (z_2 z_3) = (z_1 z_2) z_3$.

(V). Distributive law: $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$.

As an illustration, we shall show only (I). Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ then

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i = (a_2 + a_1) + (b_2 + b_1)i$$

$$= (a_2 + b_2i) + (a_1 + b_1i) = z_2 + z_1.$$

Clearly, $\mathbb{C}$ with addition and multiplication forms a field.

We also note that, for any integer $k$,

$$i^{4k} = 1, \; i^{4k+1} = i, \; i^{4k+2} = -1, \; i^{4k+3} = -i.$$ 

The rule for division is derived as

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i, \; c^2 + d^2 \neq 0.$$

**Example 1.1.** Find the quotient $\frac{(6 + 2i) - (1 + 3i)}{-1 + i - 2}$.

$$\frac{(6 + 2i) - (1 + 3i)}{-1 + i - 2} = \frac{5 - i}{-3 + i} = \frac{(5 - i)(-3 - i)}{(-3 + i)(-3 - i)}$$

$$= \frac{-15 - 1 - 5i + 3i}{9 + 1} = -\frac{8}{5} - \frac{1}{5}i.$$

Geometrically, we can represent complex numbers as points in the $xy$-plane by associating to each complex number $a + bi$ the point $(a, b)$ in the $xy$-plane (also known as an Argand diagram). The plane is referred to as the complex plane. The $x$-axis is called the real axis, and the $y$-axis is called the imaginary axis. The number $z = 0$ corresponds to the origin of the plane. This establishes a one-to-one correspondence between the set of all complex numbers and the set of all points in the complex plane.