Lecture 15
Cauchy-Goursat Theorem

In this lecture, we shall prove that the integral of an analytic function over a simple closed contour is zero. This result is one of the most fundamental theorems in complex analysis.

We begin with the following theorem from calculus.

**Theorem 15.1 (Green’s Theorem).** Let $C$ be a piecewise smooth, simple closed curve that bounds a domain $D$ in the complex plane. Let $P$ and $Q$ be two real-valued functions defined on an open set $U$ that contains $D$, and suppose that $P$ and $Q$ have continuous first order partial derivatives. Then,

$$
\int_C P\,dy - Q\,dx = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \,dxdy,
$$

where $C$ is taken along the positive direction.

The left-hand side of this equality is the line integral, whereas the right-hand side is the double integral.

Now consider a function $f(z) = u(x, y) + iv(x, y)$ that is analytic in a simply connected domain $S$. Suppose that $\gamma$ is a simple, closed, positively oriented contour lying in $S$ and given by the function $z(t) = x(t) + iy(t)$, $t \in [a, b]$. Then, we have

$$
\int_{\gamma} f(z)\,dz = \int_a^b f(z(t))\frac{dz(t)}{dt}\,dt
= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))] \left( \frac{dx}{dt} + i\frac{dy}{dt} \right) \,dt
= \int_a^b \left[ u(x(t), y(t))\frac{dx}{dt} - v(x(t), y(t))\frac{dy}{dt} \right] \,dt
+ i \int_a^b \left[ v(x(t), y(t))\frac{dx}{dt} + u(x(t), y(t))\frac{dy}{dt} \right] \,dt
= \int_{\gamma} (udx - vdy) + i \int_{\gamma} (vdx +udy).
$$

Thus, if the partial derivatives of $u$ and $v$ are continuous, then, by Green’s
Theorem, we have
\[
\int_{\gamma} f(z) \, dz = \int \int_{S'} \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \, dy + i \int \int_{S'} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \, dy,
\]
where \( S' \) is the domain interior to \( \gamma \). Since \( f(z) \) is analytic in \( S \), the first partial derivatives of \( u \) and \( v \) satisfy the Cauchy-Riemann equations. Hence, it follows that
\[
\int_{\gamma} f(z) \, dz = 0.
\]

Thus, we have shown that if \( f \) is analytic in a simply connected domain and its derivative \( f'(z) \) is continuous (recall that analyticity ensures the existence of \( f'(z) \); however, it does not guarantee the continuity of \( f'(z) \)), then its integral around any simple closed contour in the domain is zero. Goursat was the first to prove that the condition of continuity on \( f' \) can be omitted.

**Theorem 15.2 (Cauchy-Goursat Theorem).** If \( f \) is analytic in a simply connected domain \( S \) and \( \gamma \) is any simple, closed, rectifiable contour in \( S \), then \( \int_{\gamma} f(z) \, dz = 0 \).

**Proof.** The proof is divided into the following three steps.

**Step 1.** If \( \gamma \) is the boundary \( \partial \Delta \) of a triangle \( \Delta \), then \( \int_{\partial \Delta} f(z) \, dz = 0 \): We construct four smaller triangles \( \Delta_j, j = 1, 2, 3, 4 \) by joining the midpoints of the sides of \( \Delta \) by straight lines. Then, from Figure 15.1, it is clear that
\[
\int_{\partial \Delta} f(z) \, dz = \sum_{j=1}^{4} \int_{\partial \Delta_j} f(z) \, dz,
\]

which in view of the triangle inequality leads to
\[
\left| \int_{\partial \Delta} f(z) \, dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial \Delta_j} f(z) \, dz \right|.
\]