Lecture 17
Cauchy’s Integral Formula

In this lecture, we shall present Cauchy’s integral formula that expresses
the value of an analytic function at any point of a domain in terms of the
values on the boundary of this domain, and has numerous important applications. We shall also prove a result that paves the way for the Cauchy’s
integral formula for derivatives given in the next lecture.

Theorem 17.1 (Cauchy’s Integral Formula). Let \( \gamma \) be a
simple, closed, positively oriented contour. If \( f \) is analytic in some simply
connected domain \( S \) containing \( \gamma \) and \( z_0 \) is any point inside \( \gamma \), then

\[
f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \, dz.
\]

Proof. The function \( \frac{f(z)}{z - z_0} \) is analytic everywhere in \( S \) except at
the point \( z_0 \). Hence, in view of Theorem 16.1, the integral over \( \gamma \) is the same
as the integral over some small positively oriented circle \( \gamma_r : |z - z_0| = r \); i.e.,

\[
\int_{\gamma} \frac{f(z)}{z - z_0} \, dz = \int_{\gamma_r} \frac{f(z)}{z - z_0} \, dz.
\]

We write the right-hand side of the preceding equality as the sum of two
integrals as follows:

\[
\int_{\gamma_r} \frac{f(z)}{z - z_0} \, dz = \int_{\gamma_r} \frac{f(z_0)}{z - z_0} \, dz + \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz.
\]

However, since from Example 16.1

\[
\int_{\gamma_r} \frac{f(z_0)}{z - z_0} \, dz = f(z_0) \int_{\gamma_r} \frac{dz}{z - z_0} = f(z_0) 2\pi i,
\]
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it follows that

$$\int_{\gamma} \frac{f(z)}{z - z_0} \, dz = f(z_0) \, 2\pi i + \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz.$$  

The first two terms in the equation above are independent of $r$, and hence the value of the last term does not change if we allow $r \to 0$; i.e.,

$$\int_{\gamma_r} \frac{f(z)}{z - z_0} \, dz = f(z_0) \, 2\pi i + \lim_{r \to 0^+} \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz. \quad (17.2)$$

Let $M_r = \max\{|f(z) - f(z_0)| : z \in \gamma_r\}$. Since $f$ is continuous, such a finite number $M_r$ exists, and clearly, $M_r \to 0$ as $r \to 0$. Now, for $z$ on $\gamma_r$, we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \left| \frac{f(z) - f(z_0)}{r} \right| \leq \frac{M_r}{r}.$$  

Hence, from Theorem 13.1, we find

$$\left| \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq \frac{M_r}{r} L(\gamma_r) = \frac{M_r}{r} 2\pi r = 2\pi M_r,$$

which implies that

$$\lim_{r \to 0^+} \int_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz = 0.$$  

Therefore, equation (17.2) reduces to

$$\int_{\gamma} \frac{f(z)}{z - z_0} \, dz = f(z_0) \, 2\pi i,$$

which is the same as (17.1).

**Example 17.1.** Compute the integral $\int_{\gamma} \frac{e^{2z} + \sin z}{z - \pi} \, dz$, where $\gamma$ is the circle $|z - 2| = 2$ traversed once in the counterclockwise direction. Since the function $f(z) = e^{2z} + \sin z$ is analytic inside and on $\gamma$, and the point $z_0 = \pi$ lies inside $\gamma$, from (17.1) we have

$$\int_{\gamma} \frac{e^{2z} + \sin z}{z - \pi} \, dz = 2\pi \, f(\pi) = 2\pi e^{2\pi}.$$  

**Example 17.2.** Compute the integral $\int_{\gamma} \frac{\cos z + \sin z}{z^2 - 9} \, dz$ along the contour given in Figure 17.2. Clearly, the integrand fails to be analytic at the points $z = \pm 3$. However, only $z = 3$ lies inside $\gamma$. If we write $(\cos z +