Lecture 18
Cauchy’s Integral Formula for Derivatives

In this lecture, we shall show that, for an analytic function in a given domain, all the derivatives exist and are analytic. This result leads to Cauchy’s integral formula for derivatives. Next, we shall prove Morera’s Theorem, which is a converse of the Cauchy-Goursat Theorem. We shall also establish Cauchy’s inequality for the derivatives, which plays an important role in proving Liouville’s Theorem.

The arguments employed to prove Theorem 17.2 can be repeated. In fact, starting with the function

\[ H(z) = \int_{\gamma} g(\xi) (\xi - z)^2 d\xi \quad (z \text{ not on } \gamma), \tag{18.1} \]

it can be shown that \( H \) is analytic at each point not on \( \gamma \) and that

\[ H'(z) = 2 \int_{\gamma} \frac{g(\xi)}{(\xi - z)^3} d\xi \quad (z \text{ not on } \gamma), \]

which is obtained formally from (18.1) by differentiating with respect to \( z \) under the integral sign.

Now we shall apply these results to analytic functions. Suppose that \( f \) is analytic at some point \( z_0 \). Then, \( f \) is differentiable in some neighborhood \( U \) of \( z_0 \). Choose a positively oriented circle \( \gamma_r : |\xi - z_0| = r \) in \( U \).

\[ \text{Figure 18.1} \]
Cauchy’s Integral Formula for Derivatives

By Cauchy’s Integral Theorem, we have

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{\xi - z} d\xi \quad (z \text{ inside } \gamma_r), \quad (18.2) \]

and hence it follows from Theorem 17.2 that

\[ f'(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{(\xi - z)^2} d\xi \quad (z \text{ inside } \gamma_r). \quad (18.3) \]

Clearly, the right-hand side of (18.3) is a function of the form (18.1), and hence it has a derivative at each point inside \( \gamma_r \). Since the domain interior to \( \gamma_r \) is a neighborhood of \( z_0 \), \( f' \) is analytic at \( z_0 \).

We summarize these considerations in the following theorem.

**Theorem 18.1 (Differentiation of Analytic Functions).** If \( f \) is analytic in a domain \( S \), then all its derivatives \( f', f'', \ldots, f^{(n)}, \ldots \) exist and are analytic in \( S \).

**Remark 18.1.** The analogue of Theorem 18.1 for real functions does not hold, for example, the function \( f(x) = x^{3/2}, \ x \in \mathbb{R} \) is differentiable for all real \( x \), but \( f'(x) = (3/2)x^{1/2} \) has no derivative at \( x = 0 \).

Now, repeated differentiation of (18.3) with respect to \( z \) under the integral sign leads to the following result.

**Theorem 18.2 (Cauchy’s Integral Formula for Derivatives).** If \( f \) is analytic inside and on a simple, closed, positively oriented contour \( \gamma \), and if \( z \) is any point inside \( \gamma \), then

\[ f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi, \quad n = 1, 2, \ldots. \quad (18.4) \]

From an applications point of view, it is better to write (18.2) and (18.4) in the equivalent form

\[ \int_{\gamma} \frac{f(z)}{(z - z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0), \quad n = 1, 2, \ldots; \quad (18.5) \]

here, \( z_0 \) is inside \( \gamma \).

**Example 18.1.** Compute \( \int_{\gamma} \frac{\sin 3z}{z^4} dz \), where \( \gamma \) is the circle \( |z| = 1 \) traversed once counterclockwise. Since \( f(z) = \sin 3z \) is analytic inside and on \( \gamma \), from (18.5) with \( z_0 = 0 \) and \( n = 4 \), we have

\[ \int_{\gamma} \frac{\sin 3z}{z^4} dz = \frac{2\pi i}{3!} f'''(0) = -9\pi i. \]