Lecture 35
Contour Integrals Involving Multi-valued Functions

In previous lectures, we successfully applied contour integration theory to evaluate integrals of real-valued functions. However, often it turns out that the extension of a real function to the complex plane is a multi-valued function. In this lecture, we shall show that by using contours cleverly some such functions can also be integrated.

We begin with the evaluation of the integral

\[ I = \int_0^\infty x^{\alpha-1} f(x) \, dx, \quad 0 < \alpha < 1. \] (35.1)

For this, we shall assume that: (i) \( f(z) \) is a single-valued analytic function, except for a finite number of isolated singularities not on the positive real semiaxis, (ii) \( f(z) \) has a removable singularity at \( z = 0 \), and (iii) \( f(z) \) has a zero of order at least one at \( z = \infty \).

Consider the domain \( S : 0 < \arg z < 2\pi \), which is the \( z \)-plane cut along the positive real semiaxis. Clearly, the function \( F(z) = z^{\alpha-1} f(z) \) in the domain \( S \) is single-valued, its singularities are the same as those of \( f(z) \), and it coincides with \( x^{\alpha-1} f(x) \) on the upper side of the cut; i.e., \( \arg z = 0 \). Let \( \rho > 0 \) be sufficiently large so that all singularities \( z_j \) of \( f(z) \) lie inside the circle \( \gamma_\rho \), and let \( r > 0 \) be sufficiently small so that all singularities of \( f(z) \) lie outside the circle \( \gamma_r \). In the domain \( S \), we consider the closed contour \( \Gamma \), which consists of open circles \( \gamma_\rho, \gamma_r \) and the segments of the real axis \([r, \rho]\) on the upper and lower sides of the cut (see Figure 35.1). Then, by Theorem 31.3, it follows that

\[
\int_{\Gamma} F(z) \, dz = 2\pi i \sum R[z^{\alpha-1} f(z), z_j] \\
= \int_r^\rho x^{\alpha-1} f(x) \, dx + \int_{\gamma_\rho} z^{\alpha-1} f(z) \, dz \\
+ \int_{\gamma_r} z^{\alpha-1} f(z) \, dz - \int_{\gamma_r} z^{\alpha-1} f(z) \, dz \\
= I_1 + I_2 + I_3 - I_4, \tag{35.2}
\]

where the sum is taken over all singularities of \( f(z) \).
From assumption (iii) it is clear that $|f(z)| \leq M/|z|$ for all sufficiently large $|z|$. Thus, it follows that

$$|I_2| = \left| \int_{\gamma_\rho} z^{\alpha-1} f(z) dz \right| \leq \frac{M \rho^{\alpha-1}}{\rho} 2\pi \rho = 2\pi M \rho^{\alpha-1} \to 0 \text{ as } \rho \to \infty.$$ 

For the integral $I_3$, we note that $\arg z = 2\pi$, and hence $z = xe^{2\pi i}$, $x > 0$. Thus, we have

$$I_3 = \int_{\rho}^{r} z^{\alpha-1} f(z) dz = -e^{2\pi i(\alpha-1)} \int_{r}^{\rho} x^{\alpha-1} f(x) dx = -e^{2\pi i(\alpha-1)} I_1.$$ 

For the integral $I_4$, in view of assumption (ii), we find

$$|I_4| = \left| \int_{\gamma_\rho} z^{\alpha-1} f(z) dz \right| \leq C r^{\alpha-1} 2\pi r \to 0 \text{ as } r \to 0.$$ 

Finally, in (35.2), letting $r \to 0$, $\rho \to \infty$ and using the relations above, we obtain

$$\int_{0}^{\infty} x^{\alpha-1} f(x) dx = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum R[z^{\alpha-1} f(z), z_j], \quad 0 < \alpha < 1. \quad (35.3)$$

**Example 35.1.** For $0 < \alpha < 1$, from (35.3), we have

$$\int_{0}^{\infty} \frac{x^{\alpha-1}}{x+1} dx = \frac{2\pi i}{1 - e^{2\pi i\alpha}} R\left[\frac{z^{\alpha-1}}{z+1}, -1\right] = \frac{2\pi i}{1 - e^{2\pi i\alpha}} (-1)^{\alpha-1} = \frac{2\pi i e^{\pi(\alpha-1)i}}{1 - e^{2\pi(\alpha-1)i}} = \frac{\pi}{\sin \alpha \pi},$$

which is the same as given in Problem 33.7.