Lecture 37
Argument Principle and Rouché and Hurwitz Theorems

We begin this lecture with an extension of Theorem 26.3 known as the Argument Principle. This result is then used to establish Rouché’s Theorem, which provides locations of the zeros and poles of meromorphic functions. We shall also prove an interesting result due to Hurwitz.

**Theorem 37.1 (Argument Principle).** Let \( f(z) \) be meromorphic inside and on a positively oriented contour \( \gamma \). Furthermore, let \( f(z) \neq 0 \) on \( \gamma \). Then,
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = Z_f - P_f,
\]
(37.1)
holds, where \( Z_f (P_f) \) is the number of zeros (poles), counting multiplicities of \( f(z) \) that lie inside \( \gamma \).

**Proof.** Let \( a_1, \ldots, a_\ell \) be the zeros and \( b_1, \ldots, b_p \) be the poles of \( f(z) \) in \( \gamma \) with respective multiplicities \( m_1, \ldots, m_\ell \) and \( n_1, \ldots, n_p \) such that \( Z_f = m_1 + \cdots + m_\ell \) and \( P_f = n_1 + \cdots + n_p \). Then, from (26.3), we obtain
\[
R \left[ \frac{f'}{f}, a_k \right] = m_k.
\]
(37.2)
Now, similar to (26.2) at the pole \( b_s \), we have
\[
f(z) = \frac{1}{(z - b_s)^{n_s} h(z)},
\]
(37.3)
where \( h(z) \) is analytic and nonzero in a neighborhood of \( b_s \). From (37.3), it follows that
\[
\frac{f'(z)}{f(z)} = - \frac{n_s}{z - b_s} + \frac{h'(z)}{h(z)},
\]
(37.4)
and hence
\[
R \left[ \frac{f'}{f}, b_s \right] = - n_s.
\]
(37.5)
Finally, using Theorem 31.2 and the formulas (37.2) and (37.5), we find
\[
\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 2\pi i \left[ (m_1 + \cdots + m_\ell) - (n_1 + \cdots + n_p) \right] = 2\pi i [Z_f - P_f],
\]
\[R.P. \text{ Agarwal et al., An Introduction to Complex Analysis, DOI 10.1007/978-1-4614-0195-7_37, © Springer Science+Business Media, LLC 2011}\]
which is the same as (37.1).

**Remark 37.1.** The nomenclature argument principle for (37.1) comes from the fact that its left-hand side can be interpreted as the change in the argument as one runs around the image path \( f(\gamma) \). More precisely, the relation (37.1) implies that

\[
\int_{\gamma} f'(z) f(z) dz = \frac{1}{2\pi i} \left[ 2\pi i (3 + 7 + 3) - 2\pi i (4 + 7) \right] = 2 = Z_f - P_f.
\]

**Example 37.1.** Consider the function

\[
f(z) = \frac{(z - 2)^3(z - 1)^7z^3}{(z - i)^4(z + 3)^5(z - 2i)^2}
\]

and \( \gamma : |z| = 2.5 \). Clearly, \( Z_f = 3 + 7 + 3 = 13 \) and \( P_f = 4 + 7 = 11 \), and hence \( Z_f - P_f = 2 \). We also note that

\[
\frac{f'(z)}{f(z)} = \frac{3}{z - 2} + \frac{7}{z - 1} + \frac{3}{z} - \frac{4}{z - i} - \frac{5}{z + 3} - \frac{7}{z - 2i},
\]

and hence

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \left[ 2\pi i (3 + 7 + 3) - 2\pi i (4 + 7) \right] = 2 = Z_f - P_f.
\]

**Lemma 37.1.** Suppose that \( f(z) \) is continuous and assumes only integer values on a domain \( S \). Then, \( f(z) \) is a constant on \( S \).

**Proof.** For each integer \( n \), let \( S_n \subset S \) be such that, for \( z \in S_n \), \( f(z) = n \). Clearly, the sets \( S_n \) are disjoint and their union is \( S \). We claim that each \( S_n \) is open. For this, let \( z_0 \in S_n \), so that \( f(z_0) = n \). Since \( f(z) \) is continuous at \( z_0 \), there exists an open disk \( B(z_0, r) \) such that \( |f(z) - n| < 1/2 \) for all \( z \in B(z_0, r) \). However, since \( f(z) \) is integer-valued, this implies that \( f(z) = n \) for all \( z \in B(z_0, r) \). Hence, \( S_n \) is open. Now, since a connected set cannot be the union of disjoint open sets, only one of the \( S_n \) can be nonempty, say \( S_{n_0} \), and thus \( f(z) = n_0 \) for all \( z \) in \( S \).

Now we shall use Theorem 37.1 and Lemma 37.1 to prove the following result.

**Theorem 37.2 (Rouché’s Theorem).** Suppose \( f(z) \) and \( g(z) \) are meromorphic functions in a domain \( S \). If \( |f(z)| > |g(z)| \) for all \( z \) on \( \gamma \), where \( \gamma \) is a simply closed positively oriented contour in \( S \) and \( f(z) \) and \( g(z) \) have no zeros or poles on \( \gamma \), then

\[
Z_f - P_f = Z_{f+g} - P_{f+g}.
\]

**Proof.** We claim that the function \( f(z) + g(z) \) has no zeros on \( \gamma \). Indeed, if \( f(z_0) + g(z_0) = 0 \) for some \( z_0 \) on \( \gamma \), then \( |f(z_0)| = |g(z_0)| \), which contradicts