Lecture 4
Set Theory
in the Complex Plane

In this lecture, we collect some essential definitions about sets in the complex plane. These definitions will be used throughout without further mention.

The set $S$ of all points that satisfy the inequality $|z - z_0| < \epsilon$, where $\epsilon$ is a positive real number, is called an open disk centered at $z_0$ with radius $\epsilon$ and denoted as $B(z_0, \epsilon)$. It is also called the $\epsilon$-neighborhood of $z_0$, or simply a neighborhood of $z_0$. In Figure 4.1, the dashed boundary curve means that the boundary points do not belong to the set. The neighborhood $|z| < 1$ is called the open unit disk.

![Figure 4.1](image)

A point $z_0$ that lies in the set $S$ is called an interior point of $S$ if there is a neighborhood of $z_0$ that is completely contained in $S$.

**Example 4.1.** Every point $z$ in an open disk $B(z_0, \epsilon)$ is an interior point.

**Example 4.2.** If $S$ is the right half-plane $\text{Re}(z) > 0$ and $z_0 = 0.01$, then $z_0$ is an interior point of $S$.

![Figure 4.2](image)
Example 4.3. If \( S = \{ z : |z| \leq 1 \} \), then every complex number \( z \) such that \( |z| = 1 \) is not an interior point, whereas every complex number \( z \) such that \( |z| < 1 \) is an interior point.

If every point of a set \( S \) is an interior point of \( S \), we say that \( S \) is an open set. Note that the empty set and the set of all complex numbers are open, whereas a finite set of points is not open.

It is often convenient to add the element \( \infty \) to \( \mathbb{C} \). The enlarged set \( \mathbb{C} \cup \{ \infty \} \) is called the extended complex plane. Unlike the extended real line, there is no \( -\infty \). For this, we identify the complex plane with the \( xy \)-plane of \( \mathbb{R}^3 \), let \( S \) denote the sphere with radius 1 centered at the origin of \( \mathbb{R}^3 \), and call the point \( N = (0,0,1) \) on the sphere the north pole. Now, from a point \( P \) in the complex plane, we draw a line through \( N \). Then, the point \( P \) is mapped to the point \( P' \) on the surface of \( S \), where this line intersects the sphere. This is clearly a one-to-one and onto (bijective) correspondence between points on \( S \) and the extended complex plane. In fact, the open disk \( B(0,1) \) is mapped onto the southern hemisphere, the circle \( |z| = 1 \) onto the equator, the exterior \( |z| > 1 \) onto the northern hemisphere, and the north pole \( N \) corresponds to \( \infty \). Here, \( S \) is called the Riemann sphere and the correspondence is called a stereographic projection (see Figure 4.3). Thus, the sets of the form \( \{ z : |z - z_0| > r > 0 \} \) are open and called neighborhoods of \( \infty \). In what follows we shall make the following conventions: \( z_1 + \infty = \infty + z_1 = \infty \) for all \( z_1 \in \mathbb{C} \), \( z_2 \times \infty = \infty \times z_2 = \infty \) for all \( z_2 \in \mathbb{C} \) but \( z_2 \neq 0 \), \( z_1/0 = \infty \) for all \( z_1 \neq 0 \), and \( z_2/\infty = 0 \) for \( z_2 \neq \infty \).

![Figure 4.3](image_url)

A point \( z_0 \) is called an exterior point of \( S \) if there is some neighborhood of \( z_0 \) that does not contain any points of \( S \). A point \( z_0 \) is said to be a