Lecture 43
Weierstrass’s Factorization Theorem

In this lecture, we shall provide representations of entire functions as finite/infinite products involving their finite/infinite zeros. We begin with the following simple cases.

**Theorem 43.1.** If an entire function \( f(z) \) has no zeros, then \( f(z) \) is of the form \( f(z) = e^{g(z)} \), where \( g(z) \) is an entire function.

**Proof.** Clearly, the function
\[
h(z) = \frac{f'(z)}{f(z)} = \frac{d}{dz} \log f(z)
\]
is entire, and hence
\[
\int_0^z h(\xi) d\xi = \log f(\xi)|_0^z = \log f(z) - \log f(0).
\]
Thus, \( f(z) = e^{g(z)} \), where \( g(z) = \int_0^z h(\xi) d\xi + \log f(0) \).

**Theorem 43.2.** Let \( z_1, \ldots, z_m \) be the distinct zeros of an entire function \( f(z) \), where \( z_j \) is of order \( k_j \), \( j = 1, \ldots, m \). Then, \( f(z) \) is of the form
\[
f(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m} e^{g(z)}, \quad (43.1)
\]
where \( g(z) \) is an entire function.

**Proof.** It suffices to note that the function
\[
F(z) = \frac{f(z)}{(z - z_1)^{k_1} \cdots (z - z_m)^{k_m}}
\]
is entire and has no zeros. The result now follows from Theorem 43.1.

**Example 43.1.** In Theorem 43.2, if \( g(z) \) is a constant, then \( f(z) \) is a polynomial. Otherwise, it is an entire transcendental function.

**Remark 43.1.** If \( f_1(z) \) and \( f_2(z) \) are two entire functions having the same zeros with the same multiplicities, then \( f_1(z) = f_2(z) e^{g(z)} \), where \( g(z) \)
Lecture 43

is an entire function. Conversely, if $g(z)$ is an entire function, then $f(z)e^{g(z)}$ has the same zeros as $f(z)$, counting multiplicities.

Now let us assume that the entire function $f(z)$ has an infinite number of zeros, say $z_1, z_2, \ldots$, which have been ordered in increasing absolute value; i.e., $|z_1| \leq |z_2| \leq \cdots$. Clearly, $|z_n| \to \infty$, for otherwise the function $f(z)$ will be identically zero (see Corollary 26.3). In view of Examples 42.4 and 42.5, the function $f(z)$ cannot be represented as $A \prod_{j=1}^{\infty} (1 - z/z_j)$ or $A \prod_{j=1}^{\infty} (z - z_j)$. In fact, to find a proper representation of such a function, we need Weierstrass’s elementary functions:

$$E_0(z) = (1 - z), \quad E_\ell(z) = (1 - z) \exp(z + (z^2/2) + \cdots +(z^\ell/\ell)), \quad \ell = 1, 2, \cdots$$

and the following lemmas.

Lemma 43.1. Each elementary function $E_\ell(z)$, $\ell = 0, 1, 2, \cdots$ is an entire function having a simple zero at $z = 1$. Furthermore,

(i) $E_\ell'(z) = -z^\ell \exp(z + (z^2/2) + \cdots +(z^\ell/\ell))$,

(ii) if $E_\ell(z) = \sum_{j=0}^{\infty} a_j z^j$ is the power series expansion of $E_\ell(z)$ about $z = 0$, then $a_0 = 1$, $a_1 = a_2 = \cdots = a_\ell = 0$ and $a_j \leq 0$ for $j > \ell$, and

(iii) if $|z| \leq 1$, then $|E_\ell(z) - 1| \leq |z|^{\ell+1}$.

Proof. (i). Follows by a direct verification.

(ii). That $a_0 = 1$ is obvious. Since $E_\ell'(z)$ has a zero of multiplicity $\ell$ at 0, and since term-by-term differentiation is permissible, in $E_\ell(z) = \sum_{j=0}^{\infty} a_j z^j$, it follows that $a_1 = a_2 = \cdots = a_\ell = 0$. Next, in the expansion of $-E_\ell'(z) = z^\ell \exp(z + (z^2/2) + \cdots +(z^\ell/\ell))$, the coefficient of each $z^j$ is a nonnegative real number. Hence, the coefficients in the expansion of $E_\ell(z)$ must be nonpositive.

(iii). From part (ii), we have

$$|E_\ell(z) - 1| \leq \sum_{j=\ell+1}^{\infty} a_j z^j \leq \sum_{j=\ell+1}^{\infty} |a_j| |z|^j \leq |z|^{\ell+1} \sum_{j=\ell+1}^{\infty} (-a_j) |z|^{j-\ell-1}.$$

Now, since $E_\ell(1) = 0 = 1 + \sum_{j=\ell+1}^{\infty} a_j$ or $-\sum_{j=\ell+1}^{\infty} a_j = 1$, for $|z| \leq 1$ it follows that

$$|E_\ell(z) - 1| \leq |z|^{\ell+1} \left( \sum_{j=\ell+1}^{\infty} -a_j \right) \cdot 1 \leq |z|^{\ell+1}.$$

Corollary 43.1. For any nonzero $z_j$, $|E_\ell(z/z_j) - 1| \leq |z/z_j|^{\ell+1}$ for $|z| \leq |z_j|$.