Lecture 50

History of Complex Numbers

The problem of complex numbers dates back to the 1st century, when Heron of Alexandria (about 75 AD) attempted to find the volume of a frustum of a pyramid, which required computing the square root of $81 - 144$ (though negative numbers were not conceived in the Hellenistic world). We also have the following quotation from Bhaskara Acharya (working in 486 AD), a Hindu mathematician: “The square of a positive number, also that of a negative number, is positive: and the square root of a positive number is two-fold, positive and negative; there is no square root of a negative number, for a negative number is not square.” Later, around 850 AD, another Hindu mathematician, Mahavira Acharya, wrote: “As in the nature of things, a negative (quantity) is not a square (quantity), it has therefore no square root.” In 1545, the Italian mathematician, physician, gambler, and philosopher Girolamo Cardano (1501-76) published his Ars Magna (The Great Art), in which he described algebraic methods for solving cubic and quartic equations. This book was a great event in mathematics. In fact, it was the first major achievement in algebra in 3000 years, after the Babylonians showed how to solve quadratic equations. Cardano also dealt with quadratics in his book. One of the problems that he called “manifestly impossible” is the following: Divide 10 into two parts whose product is 40; i.e., find the solution of $x + y = 10$, $xy = 40$, or, equivalently, the solution of the quadratic equation $40 - x(10 - x) = x^2 - 10x + 40 = 0$, which has the roots $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Cardano formally multiplied $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ and obtained 40; however, to calculations he said “putting aside the mental tortures involved.” He did not pursue the matter but concluded that the result was “as subtle as it is useless.” This event was historic since it was the first time the square root of a negative number had been explicitly written down. For the cubic equation $x^3 = ax + b$, the so-called Cardano formula is

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}.$$  

When applied to the historic example $x^3 = 15x + 4$, the formula yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$  

Although Cardano claimed that his general formula for the solution of the cubic equation was inapplicable in this case (because of the appearance of
square roots of negative numbers could no longer be so lightly dismissed. Whereas for the quadratic equation (e.g., \( x^2 + 1 = 0 \)) one could say that no solution exists, for the cubic \( x^3 = 15x + 4 \) a real solution, namely \( x = 4 \), does exist; in fact, the two other solutions, \(-2 \pm \sqrt{3}\), are also real. It now remained to reconcile the formal and “meaningless” solution \( x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \) of \( x^3 = 15x + 4 \), found by using Cardano’s formula, with the solution \( x = 4 \), found by inspection. The task was undertaken by the hydraulic engineer Rafael Bombelli (1526-73) about thirty years after the publication of Cardano’s work.

Bombelli had the “wild thought” that since the radicals \( 2 + \sqrt{-121} \) and \( 2 - \sqrt{-121} \) differ only in sign, the same might be true of their cube roots. Thus, he let

\[
\sqrt[3]{2 + \sqrt{-121}} = a + \sqrt{-b} \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}} = a - \sqrt{-b}
\]

and proceeded to solve for \( a \) and \( b \) by manipulating these expressions according to the established rules for real variables. He deduced that \( a = 2 \) and \( b = 1 \) and thereby showed that, indeed,

\[
\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4.
\]

Bombelli had thus given meaning to the “meaningless.” This event signaled the birth of complex numbers. A breakthrough was achieved by thinking the unthinkable and daring to present it in public. Thus, the complex numbers forced themselves in connection with the solutions of cubic equations rather than the quadratic equations.

To formalize his discovery, Bombelli developed a calculus of operations with complex numbers. His rules, in our symbolism, are \((-i)(-i) = 1\) and

\[
(\pm 1)i = \pm i, \quad (+i)(+i) = -1, \quad (-i)(+i) = +1, \quad (\pm 1)(-i) = \mp i, \quad (+i)(-i) = +1.
\]

He also considered examples involving addition and multiplication of complex numbers, such as \(8i + (-5i) = 3i\) and

\[
\left( \sqrt[3]{4 + \sqrt{2i}} \right) \left( \sqrt[3]{3 + \sqrt{8i}} \right) = \sqrt[3]{8 + 11 \sqrt{2i}}.
\]

Bombelli thus laid the foundation stone of the theory of complex numbers. However, his work was only the beginning of the saga of complex numbers. Although his book *l’Algebra* was widely read, complex numbers were shrouded in mystery, little understood, and often entirely ignored. In fact, for complex numbers Simon Stevin (1548-1620) in 1585 remarked that “there is enough legitimate matter, even infinitely much, to exercise oneself without occupying oneself and wasting time on uncertainties.” John