Lecture 7
Analytic Functions II

In this lecture, we shall first prove Theorem 6.5 and then through simple examples demonstrate how easily this result can be used to check the analyticity of functions. We shall also show that the real and imaginary parts of an analytic function are solutions of the Laplace equation.

**Proof of Theorem 6.5.** From calculus, the increments of the functions $u(x,y)$ and $v(x,y)$ in the neighborhood of the point $(x_0, y_0)$ can be written as

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = u_x(x_0, y_0) \Delta x + u_y(x_0, y_0) \Delta y + \eta_1(x, y)$$

and

$$v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = v_x(x_0, y_0) \Delta x + v_y(x_0, y_0) \Delta y + \eta_2(x, y),$$

where

$$\lim_{|\Delta z| \to 0} \frac{\eta_1(x, y)}{|\Delta z|} = 0 \quad \text{and} \quad \lim_{|\Delta z| \to 0} \frac{\eta_2(x, y)}{|\Delta z|} = 0.$$

Thus, in view of the Cauchy-Riemann conditions (6.5), it follows that

$$f'(z_0) = \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u_x(x_0, y_0) \frac{\Delta x + i \Delta y}{\Delta x + i \Delta y} + v_x(x_0, y_0) \frac{i \Delta x - \Delta y}{\Delta x + i \Delta y} + \frac{\eta_1(x, y) + i \eta_2(x, y)}{\Delta x + i \Delta y} = [u_x(x_0, y_0) + iv_x(x_0, y_0)] + \frac{\eta(z)}{\Delta z},$$

where $\eta(z) = \eta_1(x, y) + i \eta_2(x, y)$. Now, taking the limit as $\Delta z \to 0$ on both sides and using the fact that $\eta(z)/\Delta z \to 0$ as $\Delta z \to 0$, we obtain

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Combining Theorems 6.4 and 6.5, we find that a necessary and sufficient condition for the analyticity of a function $f(z) = u(x, y) + iv(x, y)$ in a domain $S$ is the existence of the continuous partial derivatives $u_x, u_y, v_x,$ and $v_y$, which satisfy the Cauchy-Riemann conditions (6.5). From this it immediately follows that if $f(z)$ is analytic in a domain $S$, then the function $g(z) = \bar{f}(z)$ is not analytic in $S$. 

Example 7.1. Consider the exponential function \( f(z) = e^z = e^x (\cos y + i \sin y) \). Then, \( u(x, y) = e^x \cos y \), \( v(x, y) = e^x \sin y \), and
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x \sin y
\]
everywhere, and these derivatives are everywhere continuous. Hence, \( f'(z) \) exists and
\[
f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z = f(z).
\]

Example 7.2. The function \( f(z) = z^3 = x^3 - 3xy^2 + i(3x^2y - y^3) \) is an entire function and \( f'(z) = 3z^2 \).

Example 7.3. Consider the function \( f(z) = x^2 + y + i(2y - x) \). We have \( u(x, y) = x^2 + y \), \( v(x, y) = 2y - x \), and \( u_x = 2x \), \( u_y = 1 \), \( v_x = -1 \), \( v_y = 2 \). Thus, the Cauchy-Riemann equations are satisfied when \( x = 1 \). Since all partial derivatives of \( f \) are continuous, we conclude that \( f'(z) \) exists only on the line \( x = 1 \) and
\[
f'(1 + iy) = \frac{\partial u}{\partial x}(1, y) + i \frac{\partial v}{\partial x}(1, y) = 2 - i.
\]

Example 7.4. Let \( f(z) = z e^{-|z|^2} \). Determine the points at which \( f'(z) \) exists, and find \( f'(z) \) at these points. Since \( f(z) = (x - iy)e^{-(x^2+y^2)} \), \( u(x, y) = xe^{-(x^2+y^2)} \), \( v(x, y) = -ye^{-(x^2+y^2)} \), and
\[
\frac{\partial u}{\partial x} = e^{-(x^2+y^2)} - 2x^2e^{-(x^2+y^2)}, \quad \frac{\partial u}{\partial y} = -2xye^{-(x^2+y^2)},
\]
\[
\frac{\partial v}{\partial x} = 2xye^{-(x^2+y^2)}, \quad \frac{\partial v}{\partial y} = -e^{-(x^2+y^2)} + 2y^2e^{-(x^2+y^2)}.
\]
Thus, \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \) is always satisfied, and \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) holds, if and only if
\[
2e^{-(x^2+y^2)} - 2x^2e^{-(x^2+y^2)} - 2y^2e^{-(x^2+y^2)} = 0,
\]
or
\[
2e^{-(x^2+y^2)}(1 - x^2 - y^2) = 0,
\]
or \( x^2 + y^2 = 1 \). Since all the partial derivatives of \( f \) are continuous, we conclude that \( f'(z) \) exists on the unit circle \( |z| = 1 \). Furthermore, on \( |z| = 1 \),
\[
f'(x + iy) = \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y)
\]
\[
= e^{-(x^2+y^2)} - 2x^2e^{-(x^2+y^2)} + 2iye^{-(x^2+y^2)}
\]
\[
= e^{-1}(1 - 2x^2 + 2yi) .
\]