Chapter 14
Semigroups of Steepest Descent for Differential Equations

The first problem in this chapter seeks to make the point that for a given linear transformation \( A \) on a finite-dimensional space to itself, an adjoint for \( A \) depends on a choice of inner products, one for the domain space and one for the range space.

**Problem 204** Suppose that \( A \in L(R^2,R^2) \) defined by
\[
A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in R^2.
\]
Suppose also that in addition to the standard inner product \( \langle \cdot, \cdot \rangle_{R^2} \), one has a second inner product \( \langle \cdot, \cdot \rangle_S \) defined by
\[
\langle \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle_S = \langle \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle_{R^2} + (r - s)(u - v), \quad \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in R^2.
\]
Find a linear transformation \( B \in L(R^2,R^2) \) such that
\[
\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \rangle_{R^2} = \langle \begin{pmatrix} u \\ v \end{pmatrix}, B\begin{pmatrix} r \\ s \end{pmatrix} \rangle_S, \quad \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \in R^2.
\]

For the remainder of this chapter \( H \) denotes a Hilbert space. There are two objectives for the problems in this chapter. One is to describe an important class of semigroups. The other is to introduce a theory of steepest descent for partial differential equations.

**Problem 205** Show that if \( f \) is a continuous linear function from \( H \) to \( R \) (that is to say, a member of the dual space \( H^* \) of \( H \)), then there is a unique \( y \in H \) so that
\[
f(x) = \langle x, y \rangle_H, \quad x \in H.
\]
Definition 15 Suppose that $\phi$ is a $C^1$ function from $H \rightarrow \mathbb{R}$. The gradient of $\phi$ is the function $\nabla \phi : H \rightarrow \mathbb{R}$ so that

$$\phi'(x)h = \langle h, (\nabla \phi)(x) \rangle_H, \ x, h \in H.$$

For the rest of this chapter we suppose that the gradient $\nabla \phi$ is defined on all of $H$ and $\nabla \phi$ is locally lipschitz, that is, if $x \in H$ there is $\delta, M > 0$ such that

$$\| (\nabla \phi)(w) - (\nabla \phi)(y) \|_H \leq M \| w - y \|_H$$

if $\| w - x \|, \| y - x \| \leq \delta$.

Problem 206 Suppose that $w > 0$, $x \in H$ and $z : [0, w] \rightarrow H$ so that

$$z(0) = x, \ z'(t) = - (\nabla \phi)(z(t)), \ t \in [0, w].$$

Show that

$$(\phi(z))'(t) = -\| (\nabla \phi)(z(t)) \|^2, \ t \in [0, w].$$

Problem 207 Show that, given $x \in H$, there is a unique function $z : [0, \infty) \rightarrow H$ such that

$$z(0) = x, \ z'(t) = - (\nabla \phi)(z(t)), \ t \in [0, \infty).$$

(14.2)

Denote by $T_\phi$ the semigroup generated by (14.2), i.e., if $x \in H$ and $s \geq 0$, then

$$T_\phi(s)x = z(s),$$

where $z$ satisfies (14.2).

Problem 208 Show that if $x \in H$ and

$$u = \lim_{t \rightarrow \infty} T_\phi(t)x \text{ exists},$$

then

$$(\nabla \phi)(u) = 0.$$

The study of limits in (14.3) is very important in the theory of semigroups. For many problems in the theory of differential equations in variational form (represented by a function $\phi$) the critical points of $\phi$ are the solutions to the problem. In the notes in the last chapter there are additional references to applications. If a system of equations does not arise from a conventional variational form, one may often construct a function $\phi$ such that its zeros are solutions. This is illustrated by means of the following sequence of problems devoted to one of the simplest possible examples cast into a variational form: Find $u$ with domain $[0, 1]$ so that

$$u' - u = 0.$$