Chapter 5
Strongly Continuous Linear Semigroups

Suppose $X$ is a Banach space. Here are some problems concerning the class of linear semigroups $T$ which are strongly continuous and have the property that $|T(t)| \leq 1$, $t \geq 0$, that is, $T$ is a strongly continuous semigroup of contractions ($T$ is also called a nonexpansive semigroup). The contraction property makes our investigation a little easier but the general case of strongly continuous linear semigroups is actually an application of the contraction case. First we use a generator of $T$ in the second sense:

$$A = \{(x, y) \in X^2 : y = \lim_{t \to 0^+} \frac{1}{t} (T(t)x - x)\}. \quad (5.1)$$

For each $\lambda > 0$ denote by $I_\lambda$ the transformation so that

$$I_\lambda x = \frac{1}{\lambda} \int_0^\infty e^{-r/\lambda} T(r)x \, dr, \ x \in X. \quad (5.2)$$

**Problem 54** Show that if $\lambda > 0$, then $|I_\lambda| \leq 1$.

**Problem 55** Show that if $x \in X$, then

$$\lim_{\lambda \to 0^+} I_\lambda x = x.$$ 

**Problem 56** Show that if $x \in X$, then $I_\lambda x \in D(A)$, the domain of $A$.

**Problem 57** Show that if $\lambda > 0$ and $x \in X$, then

$$(I - \lambda A)I_\lambda x = x,$$

that is, $I - \lambda A$ is a left inverse of $I_\lambda$.

**Problem 58** Show that if $x \in D(A)$, then

$$I_\lambda (I - \lambda A)x = x,$$

that is, $(I - \lambda A)$ is also a right inverse of $I_\lambda$. 

Some help with this problem follows.

**Problem 59** Suppose \( x \in D(A) \) and define \( h : [0, \infty) \to X \) as
\[
h(t) = T(t)x, \quad t \geq 0. \tag{5.3}
\]
Show that the right derivative \( h^+ \) of \( h \) exists in all of \([0, \infty)\) and \( h^+(t) = T(t)Ax, \quad t \geq 0 \). Show also that \( h^+ \) is continuous.

**Problem 60** Show that for \( h \) as in Problem 59, \( h' \) exists on \([0, \infty)\).

**Problem 61** Show that \( h \) in Problem 59 satisfies
\[
h'(t) = T(t)Ax, \quad t \geq 0,
\]
and
\[
h'(t) = Ah(t), \quad t \geq 0 \tag{5.4}
\]
provided that \( x \in D(A) \), the domain of \( A \).

**Problem 62** Suppose that \( x \in X \) but \( x \) is not in \( D(A) \). Show that there is a sequence \( \{x_n\}_{n=1}^{\infty} \) of members of \( D(A) \), converging to \( x \) so that if \( c > 0 \), then
\[
\{T(\cdot)x_n\}_{n=1}^{\infty}
\]
converges uniformly to
\[
T(\cdot)x \tag{5.5}
\]
on \([0, c]\).

**Definition 6** The expression in (5.5) is called a generalized solution of (5.4).

**Definition 7** Suppose \( G \) is a transformation from a subset of \( X \) into \( X \). The statement that \( G \) is closed means that
\[
\{(x, Gx) : x \in D(G)\}
\]
is a closed subset of \( X \times X \).

**Problem 63** Show that if \( \lambda \geq 0 \), then \((I - \lambda A)^{-1}\) is closed and also that \( A \) is closed.

**Problem 64** Show that \( A \in L(X, X) \) if and only if \( D(A) = X \). (Use the closed graph theorem.)

**Problem 65** Suppose \( \lambda > 0, \ x \in X \) and \( m, n \) are positive integers. Show that
\[
(I_{\lambda/n})^m x = \left(\frac{n}{\lambda}\right)^m \int_0^\infty \cdots \int_0^\infty e^{-n/\lambda(s_m + \cdots + s_1)} T(s_m + \cdots + s_1)x \ ds_m \cdots ds_1 = \left(\frac{n}{\lambda}\right)^m \int_0^\infty e^{-sn/\lambda} \frac{s^{m-1}}{(m-1)!} T(s)x \ ds.
\]