Chapter 7
K-analytic Baire Spaces

Abstract In this chapter, we show that a tvs that is a Baire space and admits a countably compact resolution is metrizable, separable and complete. We prove that a linear map \( T : E \to F \) from an F-space \( E \) having a resolution \( \{ K_\alpha : \alpha \in \mathbb{N}^N \} \) into a tvs \( F \) is continuous if each restriction \( T|_{K_\alpha} \) is continuous. This theorem (due to Drewnowski) was motivated by the Arias–De Reina–Valdivia–Saxon theorem about non-Baire dense hyperplanes in Banach spaces. We provide a large class of weakly analytic metrizable and separable Baire tvs that are not analytic (clearly such spaces are necessarily not locally convex).

7.1 Baire tvs with a bounded resolution

We know already from Corollary 3.12 that a Baire lcs that is a quasi-(\( LB \))-space is a Fréchet space. Tkachuk [399] proved that if \( C_p(X) \) is K-analytic and Baire, the space \( X \) is countable and discrete. Hence a K-analytic Baire space \( C_p(X) \) is a separable Fréchet space. In fact, Tkachuk’s theorem follows from Theorem 7.1 below due to De Wilde and Sunyach [111]; see also Valdivia [421, p. 64].

Theorem 7.1 (De Wilde–Sunyach) A Baire K-analytic lcs is a separable Fréchet space.

Lutzer, van Mill and Pol [279] exhibited a countable space \( X \) (having a unique non-isolated point) such that \( C_p(X) \) is a separable, metrizable, noncomplete Baire space that is not K-analytic. In this section, we prove Theorem 7.2 from [233], which extends Theorem 7.1.

Theorem 7.2 (Kąkol–López-Pellicer) A Baire tvs \( F \) with a relatively countably compact resolution is a separable metrizable and complete tvs.

For the proof, we need two additional results from [233].

Proposition 7.1 Every Baire tvs with a bounded resolution is metrizable. Any metrizable tvs has a bounded resolution.
Proof Let $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a bounded resolution in $E$. As usual, for $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$, set $C_{n_1,n_2,\ldots,n_k} := \bigcup \{K_\beta : \beta = (m_j), n_j = m_j, j = 1, \ldots, k\}$ and $W_k := C_{n_1,n_2,\ldots,n_k}, k \in \mathbb{N}$. Then, for every neighborhood of zero $U$ in $E$, there exists $k \in \mathbb{N}$ such that $W_k \subset 2^k U$.

Indeed, otherwise there exists a neighborhood of zero $U$ in $E$ such that for every $k \in \mathbb{N}$ there exists $x_k \in W_k$ such that $2^{-k}x_k \notin U$. Since $x_k \in W_k$ for every $k \in \mathbb{N}$, there exists $\beta_k = (m^k_n) n \in \mathbb{N}^\mathbb{N}$ such that $x_k \in K_{\beta_k}, n_j = m^k_j$, for $j = 1, 2, \ldots, k$. Set $a_n = \max \{m^k_n : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$, and set $\gamma = (a_n)$. Since $\gamma \geq \beta_k$ for every $k \in \mathbb{N}$, then $K_{\beta_k} \subset K_{\gamma}$. Hence $x_k \in K_{\gamma}$ for all $k \in \mathbb{N}$. The set $K_{\gamma}$ is bounded, so $2^{-k}x_k \to 0$ in $E$, which provides a contradiction. Since $E$ is a Baire space and

$$E = \bigcup_{n_1} C_{n_1}, C_{n_1} = \bigcup_{n_2} C_{n_1,n_2}, \ldots,$$

there exist sequences $(n_k) \in \mathbb{N}^\mathbb{N}$, $(x_k)_k$ in $E$, and a sequence $(U_k)_k$ of neighborhoods of zero in $E$ such that

$$x_k \in \text{int} \overline{W_k}, \ x_k + U_k \subset \overline{W_k},$$

for all $k \in \mathbb{N}$. Choose arbitrary closed, balanced neighborhoods of zero $U$ and $V$ in $E$ such that $V + V \subset U$. Since there exists $k \in \mathbb{N}$ such that $W_k \subset 2^k V$, we note that

$$2^{-k}U_k \subset 2^{-k}\overline{W_k} - 2^{-k}x_k \subset V + V \subset U.$$

This proves that the sequence $(2^{-k}U_k)_k$ forms a countable basis of neighborhoods of zero in $E$, so $E$ is metrizable.

Now assume that $E$ is a metrizable tvs. Let $(U_n)_n$ be a countable basis of balanced neighborhoods of zero for $E$. For $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$, set $K_\alpha := \bigcap_k n_k U_k$. It is easy to see that $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a bounded resolution in $E$. □

It is known, that if $F$ is a Čech-complete space and $E$ is a completely regular Hausdorff space containing $F$ as a dense subspace, $E \setminus F$ is of first Baire category; see [343, Corollary 13.5]. Making use of Theorem 6.1, we note that if $F$ is a metric space having a compact resolution swallowing compact sets, $F$ is Čech-complete. We provide another applicable result of this type for Baire spaces $F$ admitting a certain resolution.

**Proposition 7.2** Let $E$ be a topological space that admits a weaker topology $\xi$ generated by a metric $d$. Let $F$ be a dense Baire subset of $E$ having a resolution $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ consisting of closed sets in $\xi$. Then $E \setminus F$ is of first Baire category.

**Proof** We claim that

$$\bigcap_k \text{O}(C_{n_1,n_2,\ldots,n_k}) \subset F \quad (7.1)$$