CHAPTER 2

Partitioned Matrices, Rank, and Eigenvalues

Introduction: We begin with the elementary operations on partitioned (block) matrices, followed by discussions of the inverse and rank of the sum and product of matrices. We then present four different proofs of the theorem that the products $AB$ and $BA$ of matrices $A$ and $B$ of sizes $m \times n$ and $n \times m$, respectively, have the same nonzero eigenvalues. At the end of this chapter we discuss the often-used matrix technique of continuity argument and the tool for localizing eigenvalues by means of the Geršgorin discs.

2.1 Elementary Operations of Partitioned Matrices

The manipulation of partitioned matrices is a basic tool in matrix theory. The techniques for manipulating partitioned matrices resemble those for ordinary numerical matrices. We begin by considering a $2 \times 2$ matrix

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.
$$

An application of an elementary row operation, say, adding the second row multiplied by $-3$ to the first row, can be represented by the
matrix multiplication
\[
\begin{pmatrix}
1 & -3 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
a - 3c & b - 3d \\
c & d
\end{pmatrix}.
\]

Elementary row or column operations for matrices play an important role in elementary linear algebra. These operations (Section 1.2) can be generalized to partitioned matrices as follows.

I. Interchange two block rows (columns).
II. Multiply a block row (column) from the left (right) by a non-singular matrix of appropriate size.
III. Multiply a block row (column) by a matrix from the left (right), then add it to another row (column).

Write in matrices, say, for type III elementary row operations,
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \rightarrow \begin{pmatrix}
A & B \\
C +XA & D +XB
\end{pmatrix},
\]
where $A \in \mathbb{M}_m$, $D \in \mathbb{M}_n$, and $X$ is $n \times m$. Note that $A$ is multiplied by $X$ from the left (when row operations are performed).

*Generalized elementary matrices* are those obtained by applying a single elementary operation to the identity matrix. For instance,
\[
\begin{pmatrix}
0 & I_m \\
I_n & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
I_m & 0 \\
X & I_n
\end{pmatrix}
\]
are generalized elementary matrices of type I and type III.

**Theorem 2.1** Let $G$ be the generalized elementary matrix obtained by performing an elementary row (column) operation on $I$. If that same elementary row (column) operation is performed on a block matrix $A$, then the resulting matrix is given by the product $GA$ ($AG$).

**Proof.** We show the case of $2 \times 2$ partitioned matrices. Because we deal with this type of partitioned matrix most of the time. An argument for the general case is similar.

Let $A$, $B$, $C$, and $D$ be matrices, where $A$ and $D$ are $m$- and $n$-square, respectively. Suppose we apply a type III operation, say,