GENERALIZED DISJUNCTIVE PROGRAMMING:
A FRAMEWORK FOR FORMULATION AND
ALTERNATIVE ALGORITHMS FOR
MINLP OPTIMIZATION

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Abstract. Generalized disjunctive programming (GDP) is an extension of the disjunctive programming paradigm developed by Balas. The GDP formulation involves Boolean and continuous variables that are specified in algebraic constraints, disjunctions and logic propositions, which is an alternative representation to the traditional algebraic mixed-integer programming formulation. After providing a brief review of MINLP optimization, we present an overview of GDP for the case of convex functions emphasizing the quality of continuous relaxations of alternative reformulations that include the big-M and the hull relaxation. We then review disjunctive branch and bound as well as logic-based decomposition methods that circumvent some of the limitations in traditional MINLP optimization. We next consider the case of linear GDP problems to show how a hierarchy of relaxations can be developed by performing sequential intersection of disjunctions. Finally, for the case when the GDP problem involves nonconvex functions, we propose a scheme for tightening the lower bounds for obtaining the global optimum using a combined disjunctive and spatial branch and bound search. We illustrate the application of the theoretical concepts and algorithms on several engineering and OR problems.

Key words. Disjunctive programming, Mixed-integer nonlinear programming, global optimization.

AMS(MOS) subject classifications.

1. Introduction. Mixed-integer optimization provides a framework for mathematically modeling many optimization problems that involve discrete and continuous variables. Over the last few years there has been a pronounced increase in the development of these models, particularly in process systems engineering [15, 21, 27].

Mixed-integer linear programming (MILP) methods and codes such as CPLEX, XPRESS and GUROBI have made great advances and are currently applied to increasingly larger problems. Mixed-integer nonlinear programming (MINLP) has also made significant progress as a number of codes have been developed over the last decade (e.g. DICOPT, SBB, α-ECP, Bonmin, FilMINT, BARON, etc.). Despite these advances, three basic questions still remain in this area: a) How to develop the “best” model?, b) How to improve the relaxation in these models?, c) How to solve nonconvex GDP problems to global optimality?

Motivated by the above questions, one of the trends has been to represent discrete and continuous optimization problems by models consisting of algebraic constraints, logic disjunctions and logic relations [32, 18]. The

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basic motivation in using these representations is: a) to facilitate the modeling of discrete and continuous optimization problems, b) to retain and exploit the inherent logic structure of problems to reduce the combinatorics and to improve the relaxations, and c) to improve the bounds of the global optimum in nonconvex problems.

In this paper we provide an overview of Generalized Disjunctive Programming [32], which can be regarded as a generalization of disjunctive programming [3]. In contrast to the traditional algebraic mixed-integer programming formulations, the GDP formulation involves Boolean and continuous variables that are specified in algebraic constraints, disjunctions and logic propositions. We first address the solution of GDP problems for the case of convex functions for which we consider the big-M and the hull relaxation MINLP reformulations. We then review disjunctive branch and bound as well as logic-based decomposition methods that circumvent some of the MINLP reformulations. We next consider the case of linear GDP problems to show how a hierarchy of relaxations can be developed by performing sequential intersection of disjunctions. Finally, for the case when the GDP problem involves nonconvex functions, we describe a scheme for tightening the lower bounds for obtaining the global optimum using a combined disjunctive and spatial branch and bound search. We illustrate the application of the theoretical concepts and algorithms on several engineering and OR problems.

2. Generalized disjunctive programming. The most basic form of an MINLP problem is as follows:

\[
\begin{align*}
\min \quad Z &= f(x,y) \\
\text{s.t.} \quad g_j(x,y) &\leq 0 \quad j \in J \\
\quad x &\in X \ y \in Y
\end{align*}
\]

(MINLP)

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^1 \), \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \) are differentiable functions, \( J \) is the index set of constraints, and \( x \) and \( y \) are the continuous and discrete variables, respectively. In the general case the MINLP problem will also involve nonlinear equations, which we omit here for convenience in the presentation. The set \( X \) commonly corresponds to a convex compact set, e.g. \( X = \{ x | x \in \mathbb{R}^n, Dx \leq d, x^{lo} \leq x \leq x^{up} \} \); the discrete set \( Y \) corresponds to a polyhedral set of integer points, \( Y = \{ y | y \in \mathbb{Z}^m, Ay \leq a \} \), which in most applications is restricted to 0-1 values, \( y \in \{0, 1\}^m \). In most applications of interest the objective and constraint functions \( f, g \) are linear in \( y \) (e.g. fixed cost charges and mixed-logic constraints): \( f(x,y) = c^T y + r(x) \), \( g(x,y) = By + h(x) \). The derivation of most methods for MINLP assumes that the functions \( f \) and \( g \) are convex [14].

An alternative approach for representing discrete and continuous optimization problems is by using models consisting of algebraic constraints, logic disjunctions and logic propositions [4, 32, 40, 18, 19, 22]. This approach not only facilitates the development of the models by making the