Chapter 10
Summary and discussion

Abstract We conclude this book with a summary of the previous chapters, a discussion of the presented algorithms and techniques, and an outlook on further research as well as possible extensions. We start with the summary in Section 10.1 before a discussion of the suggested methods in Section 10.2. Finally, the work ends with an overview of further research ideas in Section 10.3.

10.1 Summary

In this book, we analyzed one of the most important techniques in deterministic global optimization, namely geometric branch-and-bound methods. The prototype algorithm was presented in Chapter 2 and possible variations were discussed therein. However, throughout all these branch-and-bound approaches the main task is to calculate the required lower bounds. Therefore, we introduced the concept of the rate of convergence which leads to a general convergence theory; see again Chapter 2. The main contribution of the present work can be found in Chapter 3 where we calculated the rate of convergence for several bounding operations collected from the literature. Using these tools, we provided a general solution technique that can be applied to a wide range of global optimization problems.

We furthermore suggested several extensions of the geometric branch-and-bound prototype algorithm. In Chapter 4, we discussed an extension for multicriteria global optimization problems where again the bounding operations collected in Chapter 3 can be used. The idea of our approach was to obtain a set that consists of $\varepsilon$-Pareto optimal solutions and contains all Pareto optimal solutions. Moreover, if one only wants to find a sharp outer approximation of the set of all Pareto optimal solutions, we presented some further discarding tests based on necessary conditions for Pareto optimality in Chapter 5. Finally, an extension for mixed continuous and combinatorial optimization problems was suggested in Chapter 6.

In the third part of the present book, some applications of geometric branch-and-bound techniques were discussed. We started with a problem in computer science...
in Chapter 7, namely with the circle detection problem. We pointed out how to use deterministic global optimization to detect shapes such as lines, circles, and ellipses in images. In a number of examples it was shown that the method is very accurate. In a second application in Chapter 8 we applied the branch-and-bound method for mixed optimization problems to integrated scheduling and location problems. In some numerical examples, we succeeded in finding exact global optimal solutions in all problem instances. Finally, we solved the median line problem: we applied the geometric branch-and-bound method to find a line in the three-dimensional Euclidean space that minimizes the sum of distances between that line and some given demand points.

10.2 Discussion

As mentioned before, a good bounding operation for a given global optimization problem is the most important choice for geometric branch-and-bound methods. Although several general bounding operations can be found in Chapter 3, it of course depends on the given problem which bounding operation should be preferred. To give an example, for the ScheLoc makespan problem even the location bounding operation with a rate of convergence of $p = 1$ yields sharp bounds not only for the rectilinear norm but also for the Euclidean norm as shown in Kalsch and Drezner (2010) and Kalsch and Scholz (2010). However, in several other location problems bounding operations with a rate of convergence of $p = 1$ are not suitable because in particular the bounds for smaller boxes are not sharp enough; see, for instance, Drezner and Suzuki (2004).

Furthermore, as mentioned in Section 1.4, a d.c. decomposition is never unique. Hence, the d.c. bounding operation of course depends on the chosen decomposition. For example, in the numerical results in Chapter 3 we used a d.c. decomposition that did not yield very good results although we have a rate of convergence of $p = 2$. With the help of Lemma 1.11, it might have also been possible to derive a particular d.c. decomposition for every subbox $Y$ such that in general much sharper bounds could be found.

Moreover, in Section 3.5 we derived a general bounding operation with an arbitrary rate of convergence of $p$. However, to obtain a lower bound for a box $Y$ we have to minimize a polynomial of degree $p - 1$ over $Y$. Therefore, the general bounding operation becomes very intensive in runtimes even for $p = 3$ and $n > 2$. Thus, from the practical point of view the general bounding operation might not be suitable for $p > 2$.

To conclude the discussion about bounding operations, finally we remark that the number of iterations throughout the branch-and-bound algorithm does not only depend on the rate of convergence but also on the constant $C$ and the optimization problem; see Example 10.1. Hence, for example, a bounding operation with a rate of convergence of $p = 3$ can be worse compared to a bounding operation with a rate of convergence of $p = 2$. 