In this chapter, we develop the notion of generalized fractal string, viewed as a measure on the half-line. This is more general than the notion of fractal string considered in Chapter 1 and in the earlier work on this subject (see the notes to Chapter 1). We will use this notion in Chapter 5 to formulate the explicit formulas which will be applied throughout the remaining chapters. Besides ordinary fractal strings, generalized fractal strings enable us to deal with strings whose lengths vary continuously or whose multiplicities are nonintegral or even infinitesimal. In Section 4.2, we discuss the spectrum of a generalized fractal string, and in Section 4.3, we briefly discuss the notion of generalized fractal spray, which will be used in Chapters 9 and 11.

The conceptual difficulties associated with the notion of frequency with noninteger multiplicity led us to introduce the formalism of generalized fractal strings presented in this chapter. The flexibility of the language of measures allows us to deal in a natural way with nonintegral multiplicities, in the case of discrete measures, as in Example 4.7 and Chapter 10, and even to formalize the intuitive notion of infinitesimal multiplicity in the case of continuous measures, as in Sections 9.2 and 10.3.

In Section 4.4, we study the properties of the measure associated with a self-similar string (defined as in Chapter 2). Although Section 4.4 is of interest in its own right, it will not be used in the rest of this book and may be omitted on a first reading.
For a measure $\eta$, we denote by $|\eta|$ the total variation measure associated with $\eta$ (see, e.g., [Coh, p. 126] or [Ru2, p. 116]),

$$|\eta|(A) = \sup \left\{ \sum_{k=1}^{m} |\eta(A_k)| \right\},$$

(4.1)

where $m \geq 1$ and $\{A_k\}_{k=1}^{m}$ ranges over all finite partitions of $A$ into disjoint measurable subsets of $(0, \infty)$. Recall that $|\eta|$ is a positive measure and that $|\eta| = \eta$ if $\eta$ is positive. (See [Coh, Chapter 4] or [Ru2, Chapter 6].)

**Definition 4.1.** (i) A generalized fractal string is either a local complex or a local positive measure on $(0, \infty)$, such that

$$|\eta|(0, x_0) = 0$$

for some positive number $x_0$.

(ii) The counting function of the reciprocal lengths, or geometric counting function of $\eta$, is defined as $N_\eta(x) = \int_0^x d\eta$. If the measure $\eta$ has atoms, it is necessary to specify how the endpoint is counted. Throughout the book, we adopt the convention that $x$ is counted half; i.e.,

$$N_\eta(x) = \int_0^x d\eta := \eta(0, x) + \frac{1}{2}\eta(\{x\}).$$

(4.2)

(iii) The dimension of $\eta$, denoted $D = D_\eta$, is the abscissa of convergence of the Dirichlet integral $\zeta_\eta(\sigma) = \int_0^\infty x^{-\sigma} |\eta|(dx)$. In other words, it is the infimum of the real numbers $\sigma$ such that the improper Riemann–Lebesgue integral $\int_0^\infty x^{-\sigma} |\eta|(dx)$ converges and is finite:

$$D = D_\eta = \inf \{\sigma \in \mathbb{R} : \int_0^\infty x^{-\sigma} |\eta|(dx) < \infty\}. $$

(4.3)

(iv) The geometric zeta function is defined as the Mellin transform of $\eta$,

$$\zeta_\eta(s) = \int_0^\infty x^{-s} \eta(dx),$$

(4.4)

for $\text{Re } s > D_\eta$.

By convention, $D_\eta = \infty$ means that $x^{-\sigma}$ is not $|\eta|$-integrable for any $\sigma$, and $D_\eta = -\infty$ means that $x^{-\sigma}$ is $|\eta|$-integrable for all $\sigma$ in $\mathbb{R}$. In this last case, $\zeta_\eta$ is a holomorphic function, defined by its Dirichlet integral (4.4) on the whole complex plane. (See, for example, [Pos, Sections 2–4] or [Wid].)

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1We refer to the notes to this chapter, Section 4.5, for the precise definition of a local measure.