Chapter 13

Flow Shops, Job Shops and Open Shops (Stochastic)

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The results for stochastic flow shops, job shops, and open shops are somewhat less extensive than those for their deterministic counterparts.

This chapter focuses first on nonpreemptive static list policies, i.e., permutation schedules, for stochastic flow shops. The optimal permutation schedules often remain optimal in the class of nonpreemptive dynamic policies as well as in the class of preemptive dynamic policies. For open shops and job shops, only the classes of nonpreemptive dynamic policies and preemptive dynamic policies are considered.

The results obtained for stochastic flow shops and job shops are somewhat similar to those obtained for deterministic flow shops and job shops. Stochastic open shops are, however, very different from their deterministic counterparts.

The first section discusses stochastic flow shops with unlimited intermediate storage and jobs not subject to blocking. The second section deals with stochastic flow shops with zero intermediate storage; the jobs are subject to blocking. The third section focuses on stochastic job shops and the last section goes over stochastic open shops.
## 13.1 Stochastic Flow Shops with Unlimited Intermediate Storage

Consider two machines in series with unlimited storage between the machines and no blocking. There are $n$ jobs. The processing time of job $j$ on machine 1 is $X_{1j}$, exponentially distributed with rate $\lambda_j$. The processing time of job $j$ on machine 2 is $X_{2j}$, exponentially distributed with rate $\mu_j$. The objective is to find the nonpreemptive static list policy or permutation schedule that minimizes the expected makespan $E(C_{\text{max}})$.

Note that this problem is a stochastic counterpart of the deterministic problem $F2 \mid || C_{\text{max}}$. The deterministic two machine problem has a very simple solution. It turns out that the stochastic version with exponential processing times has a very elegant solution as well.

### Theorem 13.1.1.

Sequencing the jobs in decreasing order of $\lambda_j - \mu_j$ minimizes the expected makespan in the class of nonpreemptive static list policies, in the class of nonpreemptive dynamic policies, and in the class of preemptive dynamic policies.

**Proof.** The proof of optimality in the class of nonpreemptive static list policies is in a sense similar to the proof of optimality in the deterministic case. It is by contradiction. Suppose another sequence is optimal. Under this sequence, there must be two adjacent jobs, say job $j$ followed by job $k$, such that $\lambda_j - \mu_j < \lambda_k - \mu_k$. It suffices to show that a pairwise interchange of these two jobs reduces the expected makespan. Assume job $l$ precedes job $j$ and let $C_{1l}$ ($C_{2l}$) denote the (random) completion time of job $l$ on machine 1 (2). Let $D_l = C_{2l} - C_{1l}$.

Perform an adjacent pairwise interchange on jobs $j$ and $k$. Let $C_{1k}$ and $C_{2k}$ denote the completion times of job $k$ on the two machines under the original, supposedly optimal, sequence and let $C'_{1j}$ and $C'_{2j}$ denote the completion times of job $j$ under the schedule obtained after the pairwise interchange. Let $m$ denote the job following job $k$. Clearly, the pairwise interchange does not affect the starting time of job $m$ on machine 1 as this starting time is equal to $C_{1k} = C'_{1j} = C_{1l} + X_{1j} + X_{1k}$. Consider the random variables

$$D_k = C_{2k} - C_{1k}$$

and

$$D'_j = C'_{2j} - C'_{1j}.$$  

Clearly, $C_{1k} + D_k$ is the time at which machine 2 becomes available for job $m$ under the original schedule, while $C_{1k} + D'_j$ is the corresponding time after the pairwise interchange. First it is shown that the random variable $D'_j$ is stochastically smaller than the random variable $D_k$. If $D_l \geq X_{1j} + X_{1k}$, then clearly $D_k = D'_j$. The case $D_l \leq X_{1j} + X_{1k}$ is slightly more complicated. Now

$$P(D_k > t \mid D_l \leq X_{1j} + X_{1k}) = \frac{\mu_j}{\lambda_k + \mu_j} e^{-\mu_k t}$$