Single machine models are important for various reasons. The single machine environment is very simple and a special case of all other environments. Single machine models often have properties that neither machines in parallel nor machines in series have. The results that can be obtained for single machine models not only provide insights into the single machine environment, they also provide a basis for heuristics that are applicable to more complicated machine environments. In practice, scheduling problems in more complicated machine environments are often decomposed into subproblems that deal with single machines. For example, a complicated machine environment with a single bottleneck may give rise to a single machine model.

In this chapter various single machine models are analyzed in detail. The total weighted completion time objective is considered first, followed by several due date related objectives, including the maximum lateness, the number of tardy jobs, the total tardiness and the total weighted tardiness. All objective functions considered in this chapter are regular.

In most models considered in this chapter there is no advantage in having preemptions; for these models it can be shown that the optimal schedule in the class of preemptive schedules is nonpreemptive. However, if jobs are released at different points in time, then it may be advantageous to preempt. If jobs are released at different points in time in a nonpreemptive environment, then
it may be advantageous to allow for unforced idleness (i.e., an optimal schedule may not be non-delay).

### 3.1 The Total Weighted Completion Time

The first objective to be considered is the total weighted completion time, i.e., $\sum w_j C_j$. The weight $w_j$ of job $j$ may be regarded as an importance factor; it may represent either a holding cost per unit time or the value already added to job $j$. This problem gives rise to one of the better known rules in scheduling theory, the so-called Weighted Shortest Processing Time first (WSPT) rule. According to this rule the jobs are ordered in decreasing order of $w_j/p_j$.

**Theorem 3.1.1.** The WSPT rule is optimal for $\sum w_j C_j$.

**Proof.** By contradiction. Suppose a schedule $S$, that is not WSPT, is optimal. In this schedule there must be at least two adjacent jobs, say job $j$ followed by job $k$, such that $\frac{w_j}{p_j} < \frac{w_k}{p_k}$.

Assume job $j$ starts its processing at time $t$. Perform a so-called **Adjacent Pair-wise Interchange** on jobs $j$ and $k$. Call the new schedule $S'$. While under the original schedule $S$ job $j$ starts its processing at time $t$ and is followed by job $k$, under the new schedule $S'$ job $k$ starts its processing at time $t$ and is followed by job $j$. All other jobs remain in their original position. The total weighted completion time of the jobs processed before jobs $j$ and $k$ is not affected by the interchange. Neither is the total weighted completion time of the jobs processed after jobs $j$ and $k$. Thus the difference in the values of the objectives under schedules $S$ and $S'$ is due only to jobs $j$ and $k$ (see Figure 3.1). Under $S$ the total weighted completion time of jobs $j$ and $k$ is

\[(t + p_j)w_j + (t + p_j + p_k)w_k,\]

while under $S'$ it is

\[(t + p_k)w_k + (t + p_k + p_j)w_j.\]

It is easily verified that if $w_j/p_j < w_k/p_k$ the sum of the two weighted completion times under $S'$ is strictly less than under $S$. This contradicts the optimality of $S$ and completes the proof of the theorem. \(\Box\)

The computation time needed to order the jobs according to WSPT is the time required to sort the jobs according to the ratio of the two parameters. A simple sort can be done in $O(n \log(n))$ time, see Example D.1.1 in Appendix D.

How is the minimization of the total weighted completion time affected by precedence constraints? Consider the simplest form of precedence constraints, i.e., precedence constraints that take the form of parallel chains (see Figure 3.2).