Chapter 4
The Degenerate Case

Recall that in our notation, a $\mu$-homeomorphism in a domain $D$, $D \subset \mathbb{C}$ is an ACL homeomorphic solution of (B) in $D$; see Sect. 1.5. For some functions $\mu$ with $|\mu(z)| \leq 1$ a.e. and $||\mu||_{\infty} = 1$, there are no $\mu$-homeomorphisms, i.e., homeomorphic solutions of (B), as illustrated below in Sects. 4.1.1 and 4.1.2. Even when a $\mu$-homeomorphism exists, it is not known whether it is unique and generates the set of all elementary solutions. As in the classical case, by an elementary solution, we mean an open and discrete solution.

Contrary to the classical case where every ACL solution belongs to $W^{1,2}_{\text{loc}}$, even to $W^{1,p}_{\text{loc}}$ with some $p > 2$, in the relaxed classical case, an ACL solution need not be in $W^{1,2}_{\text{loc}}$. Recall that the $W^{1,1}_{\text{loc}}$ property implies the ACL property but not vice versa.

The main problems in the theory of $\mu$-homeomorphisms are as in the classical case; see (3.1.1).

Approximation theorems like Theorem 2.1 and Corollary 2.3 in Chap. 2 are crucial in many existence proofs.

4.1 Examples

The following two examples show that the assumption $K \in L^p$ for some $p \geq 1$, or even that $K$ belongs to $L^p$, for all $1 \leq p < \infty$, does not guarantee the existence of $\mu$-homeomorphisms.

4.1.1 Example One

Let

$$\mu(re^{i\theta}) = -\frac{e^{2i\theta}}{1 + 2r}$$
in $D = \{ |z| < 1 \}$. Then
\[ K = 1 + \frac{1}{r} \in L^1(D). \]

We shall show that (B) has no $\mu$-homeomorphic solution in $D$. Indeed, consider the mapping
\[ g(re^{i\theta}) = (1 + r)e^{i\theta}. \]

It is easy to see that $g$ is a $\mu$-homeomorphism of $D \setminus \{0\}$ onto $1 < |w| < 2$.

Suppose that (B) has a homeomorphic solution $f : D \to \mathbb{C}$. Then, since $f$ and $g$ are locally quasiconformal in $D \setminus \{0\}$, it follows by the classical uniqueness theorem that $h = g \circ f^{-1}$ is conformal in $f(D) \setminus f(0)$ and thus can be extended to $f(D)$, which is impossible.

### 4.1.2 Example Two

Let
\[ \mu(re^{i\theta}) = -e^{2i\theta} \frac{1 + \frac{1}{\log 1/r} - \frac{1}{\log^2 1/r}}{1 + \frac{1}{\log 1/r} + \frac{1}{\log^2 1/r}}. \]
in $D(1/e) = \{ z \in \mathbb{C} : |z| < 1/e \}$. Then
\[ K = \log^2 \frac{1}{r} + \log \frac{1}{r} \in \bigcap_{1 \leq p < \infty} L^p(D(1/e)). \]

We shall show that (B) has no $\mu$-homeomorphic solution in $D(1/e)$.

Indeed, consider the mapping
\[ g(re^{i\theta}) = \left( 1 + \frac{1}{\log \frac{1}{r}} \right) e^{i\theta}. \]

It is easy to see that $g$ is a $\mu$-homeomorphism of $D(1/e) \setminus \{0\}$ onto $1 < |w| < 2$.

Then the assertion follows as in the previous example.

**Remark 4.1.** Contrary to the classical case, in the relaxed classical case as well as in the alternating case, the existence and nature of a solution may depend not only on the behavior of $|\mu|$ but also on $\arg \mu$. For instance, let $\mu$ be as in Sect. 4.1.1. Then (B) has no homeomorphic solutions in $D$. However, if $\mu$ is replaced by $\tilde{\mu} = -\mu$, then (B) with $\tilde{\mu}$ instead of $\mu$ has a solution $f(re^{i\theta}) = re^{1-1/r}e^{i\theta}$ which maps $D$ homeomorphically onto itself.