Chapter 3
Some Spectral Problems of Mechanics

3.1 Introduction

We obtain a spectral problem by formally considering a solution \( u \) of the form

\[ u(x, t) = e^{i\omega t}v(x) \]

for a dynamic equation

\[ B(u(x, t)) = \rho \frac{\partial^2 u(x, t)}{\partial t^2} \]

where \( B \) is a differential operator having coefficients independent of \( t \), defined by the model of an elastic body, and \( \rho \) is the density of the body. The spectral equation is then

\[ B(u(x, t)) = -\rho \omega^2 u(x, t) . \quad (3.1.1) \]

This equation should be supplemented with boundary conditions corresponding to those for the original dynamics problem. As in the traditional eigenvalue problems of linear algebra, we seek the set of values \( \omega_k \) such that for \( \omega = \omega_k \) there is a nontrivial solution of (3.1.1) that satisfies homogeneous boundary equations. Then \( \omega_k \) is called an eigenfrequency of the problem and the corresponding solution is an eigenfunction or eigenvector.

In this chapter we establish some properties of the spectral problems of linear mechanics. In particular, we show that for various models of bounded elastic bodies, including the membrane, all the \( \omega_k^2 \) are positive (there can be a few \( \omega_k = 0 \) that correspond to the motions of a body as a free rigid body) and the spectrum is discrete. We also establish completeness of the corresponding sets of eigenvectors, a property necessary for application of the separation of variables method to boundary value problems.
3.2 Adjoint Operator

This notion will be introduced for operators acting in a Hilbert space, although it can also be applied in other settings, as when we consider problems of linear continuum mechanics.

Let $H$ be a Hilbert space and $A$ a continuous linear operator acting in $H$. Consider the inner product $(Ax, y)$ as a functional with respect to the variable $x \in H$ when $y \in H$ is arbitrary but fixed. This functional, thanks to the linearity of $A$, is linear and bounded because

$$|(Ax, y)| \leq \|Ax\| \|y\| \leq (\|A\| \|y\|) \|x\| .$$

By the Riesz representation theorem, it can be represented in the form

$$(Ax, y) = (x, z)$$

where the element $z$ is uniquely defined by $y$ and $A$. So the correspondence $y \mapsto z$ can be viewed as an operator $A^*$, $z = A^*y$, and we call $A^*$ the adjoint of $A$.

**Lemma 3.2.1.** The adjoint $A^*$ is a linear operator.

**Proof.** By definition we get

$$(Ax, y_1) = (x, A^*y_1) , \quad (Ax, y_2) = (x, A^*y_2) ,$$

and

$$(Ax, \alpha_1y_1 + \alpha_2y_2) = (x, A^*(\alpha_1y_1 + \alpha_2y_2)) .$$

But

$$(Ax, \alpha_1y_1 + \alpha_2y_2) = \overline{\alpha_1}(Ax, y_1) + \overline{\alpha_2}(Ax, y_2) ,$$

so

$$(x, A^*(\alpha_1y_1 + \alpha_2y_2)) = \overline{\alpha_1}(x, A^*y_1) + \overline{\alpha_2}(x, A^*y_2) = (x, \alpha_1A^*y_1) + (x, \alpha_2A^*y_2) .$$

Since $x$ is an arbitrary element of $H$, we have

$$A^*(\alpha_1y_1 + \alpha_2y_2) = \alpha_1A^*y_1 + \alpha_2A^*y_2 .$$

This completes the proof. □

**Lemma 3.2.2.** We have

(i) $(A + B)^* = A^* + B^*$,
(ii) $(AB)^* = B^*A^*$.

**Proof.** Property (i) is evident. Comparing the equalities

$$(AB)x, y = (x, (AB)^*y)$$

with the definition of adjoint, we get

$$(AB)x, y = (x, B^*A^*)y .$$

But

$$(AB)x, y = (x, (AB)^*y) ,$$

and

$$(B^*A^*)y = (B^*A^*)y ,$$

so

$$(AB)x, y = (x, (AB)^*y) .$$

This completes the proof. □