Chapter 7

Loop-Erased Walk

7.1 Introduction

In this chapter we discuss the loop-erased or Laplacian self-avoiding random walk. We will primarily use the loop-erased characterization of the walk because it is the one that allows for rigorous analysis of the model. In Proposition 7.3.1 we show that this is the same as the Laplacian random walk defined in Section 6.5.

The loop-erased walk, like the usual self-avoiding walk, has a critical dimension of four. If \( d > 4 \), then the number of points remaining after erasing loops is a positive fraction of the total number of points. We prove a strong law for this fraction and show that the loop-erased process approaches a Brownian motion. If \( d \leq 4 \), the proportion of points remaining after erasing loops goes to zero. In the critical dimension \( d = 4 \), however, we can still prove a weak law for the number of points remaining. From this we can show that the process approaches a Brownian motion for \( d = 4 \), although a logarithmic correction to scaling is needed. For \( d < 4 \), the number of points erased is not uniform from path to path, and we do not expect a Gaussian limit.

Sections 7.2-7.4 give the basic properties of the loop-erased walk. The definition is a little more complicated in two dimensions because simple random walk is recurrent. In Section 7.5 we give upper bounds on the number of points remaining for \( d \leq 4 \). This allows us to give lower bounds on the mean square displacement in two and three dimensions, Theorem 7.6.2. The final section discusses the walk for \( d \geq 4 \) and proves the convergence to Brownian motion.
CHAPTER 7. LOOP-ERASED WALK

7.2 Erasing Loops

In this section we describe a procedure which assigns to each finite simple random walk path $\lambda$ a self-avoiding walk $L\lambda$. Let $\lambda = [\lambda(0), \ldots, \lambda(m)]$ be a simple random walk path of length $m$. If $\lambda$ is self-avoiding, let $L\lambda = \lambda$. Otherwise, let

$$t = \inf\{j : \lambda(i) = \lambda(j) \text{ for some } 0 \leq i < j\},$$
$$s = \text{the } i < t \text{ with } \lambda(i) = \lambda(t),$$

and let $\tilde{\lambda}$ be the $m - (t - s)$ step path

$$\tilde{\lambda}(j) = \begin{cases} 
\lambda(j), & 0 \leq j \leq s, \\
\lambda(j + t - s), & s \leq j \leq m - (t - s).
\end{cases}$$

If $\tilde{\lambda}$ is self-avoiding we let $L\lambda = \tilde{\lambda}$. Otherwise, we perform this procedure on $\tilde{\lambda}$ and continue until we eventually obtain a self-avoiding walk $L\lambda$ of length $n \leq m$. This walk clearly satisfies $(L\lambda)(0) = \lambda(0)$ and $(L\lambda)(n) = \lambda(m)$.

There is another way to define $L\lambda$ which can easily be seen to be equivalent. Let

$$s_0 = \sup\{j : \lambda(j) = \lambda(0)\},$$

and for $i > 0$,

$$s_i = \sup\{j : \lambda(j) = \lambda(s_{i-1} + 1)\}.$$

Let

$$n = \inf\{i : s_i = m\}.$$

Then

$$L\lambda = [\lambda(s_0), \lambda(s_1), \ldots, \lambda(s_n)].$$

The loop-erasing procedure depends on the order of the points. Suppose we wish to erase loops in the reverse direction. More precisely, let

$$\lambda_R(j) = \lambda(m - j), \quad 0 \leq j \leq m,$$

and define reverse loop-erasing $L^R$ by

$$L^R\lambda = (L\lambda_R)_R.$$

It is not difficult to construct $\lambda$ such that $L\lambda \neq L^R\lambda$. However, we prove here that if $\lambda$ is chosen using the distribution of simple random walk, then $L\lambda$ and $L^R\lambda$ give the same distribution. Recall that $\Lambda_m$ is the set of simple random walk paths of length $m$ starting at the origin.