The dynamic analysis using the stiffness matrix method for structures modeled as beams was presented in Chapter 10. This method of analysis when applied to beams requires the calculation of element matrices (stiffness, mass, and damping matrices), the assemblage from these matrices of the corresponding system matrices, the formation of the force vector, and the solution of the resultant equations of motion. These equations, as we have seen, may be solved in general by the modal superposition method or by numerical integration of the differential equations of motion. In this chapter and in the following chapters, the dynamic analysis of structures modeled as frames is presented.

We begin in the present chapter with the analysis of structures modeled as plane frames and with the loads acting in the plane of the frame. The dynamic analysis of such structures requires the inclusion of the axial effects in the stiffness and mass matrices. It also requires a coordinate transformation of the nodal coordinates from element or local coordinates to system or global coordinates. Except for the consideration of axial effects and the need to transform these coordinates, the dynamic analysis by the stiffness method when applied to frames is identical to the analysis of beams as discussed in Chapter 10.

11.1 Element Stiffness Matrix for Axial Effects

The inclusion of axial forces in the stiffness matrix of a flexural beam element requires the determination of the stiffness coefficients for axial loads. To derive the stiffness matrix for an axially loaded member, consider in Fig. 11.1 a beam segment
acted on by the axial forces $P_1$ and $P_2$ producing axial displacements $\delta_1$ and $\delta_2$ at the nodes of the element. For a prismatic and uniform beam segment of length $L$ and cross-sectional $A$, it is relatively simple to obtain the stiffness relation for axial effects by the application of Hooke's law. In relation to the beam shown in Fig. 11.1, the displacements $\delta_1$ produced by the force $P_1$ acting at node 1 while node 2 is maintained fixed ($\delta_2 = 0$) is given by

$$\delta_1 = \frac{P_1 L}{AE}$$

From eq. (11.1) and the definition of the stiffness coefficient $k_{11}$ (force at node 1 to produce a unit displacement, $\delta_1 = 1$), we obtain

$$k_{11} = \frac{P_1}{\delta_1} = \frac{AE}{L}$$

The equilibrium of the beam segment acted upon by the force $k_{11}$ requires a force $k_{21}$ at the other end of equal magnitude but in opposite direction, namely

$$k_{21} = -k_{11} = -\frac{AE}{L}$$

Analogously, the other stiffness coefficients due to a unit displacement at node 2 ($\delta_2 = 1$) are:

$$k_{22} = \frac{AE}{L}$$

and

$$k_{12} = -\frac{AE}{L}$$

The stiffness coefficients as given by eqs. (11.2) are the elements of the stiffness matrix relating axial forces and displacements for a prismatic beam segment, that is,

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} AE & 0 \\ 0 & AE \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

The stiffness matrix corresponding to the nodal coordinates for the beam segment shown in Fig. 11.2 is obtained by combining in a single matrix the stiffness matrix for axial effects, eq. (11.3), and the stiffness matrix for flexural effects, eq. (10.20). The matrix resulting from this combination relates the forces $P_i$ and the displacements $\delta_1$ at the coordinates indicated in Fig. 11.2 as

Fig. 11.1 Beam element showing nodal axial loads $P_1$, $P_2$, and corresponding nodal displacements $\delta_1$, $\delta_2$. 