4

BASIC ELEMENTS OF DEMPSTER–SHAFER THEORY

4.1 FROM INTUITION TO COMPATIBILITY RELATIONS

The greatest part of works dealing with the fundamentals of Dempster–Shafer theory is conceived either on the combinatoric, or on the axiomatic, but in both the cases on a very abstract level. The first approach begins by the assumption that \( S \) is a nonempty finite set, that \( m \) is a mapping which ascribes to each \( A \subseteq S \) a real number \( m(A) \) from the unit interval \([0,1]\) in such a way that \( \sum_{A \subseteq S} m(A) = 1 \) (\( m \) is called a basic probability assignment on \( S \)), and that the (normalized) belief function induced by \( m \) is the mapping \( \text{bel}_m : \mathcal{P}(S) \to [0,1] \) defined, for each \( A \subseteq S \), by \( \text{bel}_m(A) = (1 - m(\emptyset))^{-1} \sum_{\emptyset \neq B \subset A} m(B) \), if \( m(\emptyset) < 1 \), \( \text{bel}_m \) being undefined otherwise Shafer (1976) and elsewhere. The other (axiomatic) approach begins with the idea that belief function on a finite nonempty set \( S \) is a mapping \( \text{bel} : \mathcal{P}(S) \to [0,1] \), satisfying certain conditions (obeying certain axioms, in other terms). If these conditions (axioms) are strong and reasonable enough, it can be proved that it is possible to define uniquely a basic probability assignment \( m \) on \( S \) such that the belief function induced by \( m \) is identical with the original belief function defined by axioms, so that both the approaches meet each other and yield the same notion of belief function (Smets (1994)). The problems how to understand and obtain the probability distribution \( m \) over \( \mathcal{P}(S) \) in the first case, or how to justify the particular choice of the demands imposed to belief functions in the other case,
are put aside or are "picked before brackets" and they are not taken as a part of Dempster–Shafer theory in its formalized setting.

The basic stone of the probabilistic approach to Dempster–Shafer theory consists in a definition and interpretation of belief functions, as the basic quantitative characteristic of uncertainty in this theory, using appropriate terms and tools of probability theory. Like as in the more general case introduced above, we shall begin with some intuitive interpretation of the presented notions, putting this interpretation aside and returning to a purely mathematical formalized style of explanation as quickly as possible.

So, let $S$ be a nonempty, but not necessary finite, set of all possible internal states of an investigated system SYST. As in the particular case of decision making under uncertainty explained in the closing part of Chapter 3, our aim is not to optimize a general statistical decision function with respect to a given loss function and to a global decision strategy (the minimax, a Bayes, or another one), but rather to identify the actual internal state $s_0$ of the system SYST or at least to decide, whether $s_0 \in A$ holds or does not hold for a (proper, to avoid the trivial case) subset $A$ of $S$. The hidden assumption behind such a simplification is that if the decision about the internal state of SYST is correct, also the consecutive activity of the subject concerning in her/his intervention into the system will be the best possible. More generally, the better is the decision about the actual internal state of SYST, the better will be the consecutive operation executed by the subject.

Again, as in the general case above, the subject is not supposed to be able to observe immediately the actual state of SYST or to draw this information simply and beyond any risk of error from her/his knowledge concerning the system and its environment. The only what the subject knows are the results of some observations, measurements, experiments, etc., cumulated into a value $x$ from a nonempty (and possibly vector) space $E$. At the general level the finiteness of $E$ also need not be assumed. However, in order to be sure that there is at least some degree of sensefulness and rationality when taking some decisions concerning the actual state of SYST on the ground of empirical values from $E$, some relation between the states from $S$ and values from $E$ must be assumed to exist and to be known to the subject. In the case of statisti-