Wave Front Propagation for KPP-Type Equations

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1.1. INTRODUCTION

The following equation was considered in [15]:

\[
\frac{\partial u(t,x)}{\partial t} = \frac{D}{2} \frac{\partial^2 u}{\partial x^2} + f(u), \quad t > 0, \ x \in \mathbb{R},
\]

\[
\left\{ \begin{array}{ll}
0, & x < 0 \\
1, & x > 0.
\end{array} \right.
\]

Here \( D > 0 \) and \( f(u) = c(u)u \), where the function \( c(u) \) is supposed to be Lipschitz continuous, positive for \( u < 1 \) and negative for \( u > 1 \), and such that \( c = c(0) = \max_{0 \leq u \leq 1} c(u) \). Let us denote the class of such functions \( f(u) \) by \( F_1 \).

It is easy to see from (1.1.1) that \( u(t,x) \) for each \( t \geq 0 \) is a strictly monotone function decreasing from 1 as \( x \to -\infty \) to 0 as \( x \to \infty \). Thus there exists a unique \( m(t), t > 0, \) such that \( u(t,m(t)) = 1/2 \). It was proved in [15] that \( \lim_{t \to \infty} t^{-1} m(t) = \sqrt{2cD} \), and that \( u(t,m(t) + z) \to v(z) \) as \( t \to \infty \), where the
function \( v(z) \), \(-\infty < z < \infty \), is the solution of the problem

\[
\frac{D}{2} v''(z) + \alpha v'(z) + f(v(z)) = 0, \quad -\infty < z < \infty,
\]

\[
\lim_{z \to -\infty} v(z) = 0, \quad \lim_{z \to \infty} v(z) = 1, \quad v(0) = \frac{1}{2}
\]  

(1.1.2)

for \( \alpha = \sqrt{2cD} \). Problem (1.1.2) is solvable for \( \alpha \geq \sqrt{2cD} \), and the solution is unique. Roughly speaking it means that the solution of problem (1.1.1) behaves for large \( t \) as a running wave \( v(x - \alpha t) \). It can be characterized by the shape \( v(z) \) and by the speed \( \alpha = \sqrt{2cD} \).

One can introduce the asymptotic speed independently of the shape:

The number \( \alpha^* \) is called the asymptotic speed as \( t \to \infty \) for the problem (1.1.1) if for any \( h > 0 \)

\[
\lim_{t \to \infty} \sup_{x > (\alpha^* + h)t} u(t, x) = 0, \quad \lim_{t \to \infty} \inf_{x < (\alpha^* - h)t} u(t, x) = 1.
\]

It follows from [15] that such \( \alpha^* \) exists and is equal to \( \sqrt{2cD} \). The notion of asymptotic speed can be introduced in a similar way in a more general situation.

Consider a tube \( R^1 \times G \), where \( G \) is a bounded domain in \( R^r \) with a smooth boundary \( \partial G \), and the following problem:

\[
\frac{\partial u(t, x, y)}{\partial t} = \frac{D}{2} \Delta_{x,y} u - b \frac{\partial u}{\partial x} + f(u), \quad t > 0, \ x \in R^1, \ y \in G
\]

\[
\frac{\partial u(t, x, y)}{\partial n} \bigg|_{t>0,x\in R^1,y\in \partial G} = 0, \quad u(0, x, y) = \chi^-(x).
\]  

(1.1.3)

Here \( n = n(y) \) is the outward normal to \( \partial G \) at point \( y \in \partial G \); \( \Delta_{x,y} \) is the Laplacian in \( x \) and \( y \); \( f \in F_1 \) as before; \( D \) is positive.

As in the one-dimensional case \( \alpha^* \) is called the asymptotic speed as \( t \to \infty \) if for any \( h > 0 \)

\[
\lim_{t \to \infty} \sup_{x > (\alpha^* + h)t, y \in G \cup \partial G} u(t, x, y) = 0, \quad \lim_{t \to \infty} \inf_{x < (\alpha^* - h)t, y \in G \cup \partial G} u(t, x, y) = 1.
\]

Equation (1.1.3) describes the evolution of particles which diffuse with diffusivity \( D \) in the flow having velocity \( b \) and take part in the "chemical reaction" governed by the nonlinear term \( f(u) \). Of course, one cannot expect that some asymptotic speed will be established if \( D \) or \( b \) depends on \( x \) arbitrarily. Let \( D = \text{const} \) and \( b \) be independent of \( x \). If \( b = \text{const} \) then it follows from the results of [15] that the asymptotic speed for problem (1.1.3) is equal to \( \alpha^* = b + \sqrt{2cD} \). Now let the velocity of the flow \( b \) depend on the point of the cross section of the tube: \( b = b(y) \). In the linear case \( f(u) \equiv 0 \) one can check that

\[
\alpha^* = \bar{b} = \frac{1}{|G|} \int_G b(y) \, dy
\]

where \( |G| \) is the volume of the domain \( G \). The last statement is the result of averaging in the \( y \)-variables: the uniform distribution is the invariant measure for the diffusion governed by \( (D/2)\Delta_y \) in \( G \) with the normal reflection on the boundary.