Wave Front Propagation for KPP-Type Equations

Mark Freidlin

1.1. INTRODUCTION

The following equation was considered in [15]:

\[
\frac{\partial u(t, x)}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u), \quad t > 0, \ x \in R^1,
\]

\[
u(0, x) = \chi^-(x) = \begin{cases} 
1, & x \leq 0 \\
0, & x > 0.
\end{cases}
\] (1.1.1)

Here \( D > 0 \) and \( f(u) = c(u)u \), where the function \( c(u) \) is supposed to be Lipschitz continuous, positive for \( u < 1 \) and negative for \( u > 1 \), and such that \( c = c(0) = \max_{0 \leq u \leq 1} c(u) \). Let us denote the class of such functions \( f(u) \) by \( F_1 \).

It is easy to see from (1.1.1) that \( u(t, x) \) for each \( t \geq 0 \) is a strictly monotone function decreasing from 1 as \( x \to -\infty \) to 0 as \( x \to \infty \). Thus there exists a unique \( m(t), t > 0 \), such that \( u(t, m(t)) = 1/2 \). It was proved in [15] that \( \lim_{t \to \infty} t^{-1} m(t) = \sqrt{2cD} \), and that \( u(t, m(t) + z) \to v(z) \) as \( t \to \infty \), where the
function $v(z), -\infty < z < \infty$, is the solution of the problem

$$\frac{D}{2} v''(z) + \alpha v'(z) + f(v(z)) = 0, \quad -\infty < z < \infty,$$

$$\lim_{z \to \infty} v(z) = 0, \quad \lim_{z \to -\infty} v(z) = 1, \quad v(0) = \frac{1}{2} \quad (1.1.2)$$

for $\alpha = \sqrt{2cD}$. Problem (1.1.2) is solvable for $\alpha \geq \sqrt{2cD}$, and the solution is unique. Roughly speaking it means that the solution of problem (1.1.1) behaves for large $t$ as a running wave $v(x - \alpha t)$. It can be characterized by the shape $v(z)$ and by the speed $\alpha = \sqrt{2cD}$.

One can introduce the asymptotic speed independently of the shape:

The number $\alpha^*$ is called the asymptotic speed as $t \to \infty$ for the problem (1.1.1) if for any $h > 0$

$$\lim_{t \to \infty} \sup_{x > (\alpha^* + h)t} u(t, x) = 0, \quad \lim_{t \to \infty} \inf_{x < (\alpha^* - h)t} u(t, x) = 1.$$

It follows from [15] that such $\alpha^*$ exists and is equal to $\sqrt{2cD}$. The notion of asymptotic speed can be introduced in a similar way in a more general situation.

Consider a tube $R^1 \times G$, where $G$ is a bounded domain in $R^r$ with a smooth boundary $\partial G$, and the following problem:

$$\frac{\partial u(t, x, y)}{\partial t} = \frac{D}{2} \Delta_{x, y} u - b \frac{\partial u}{\partial x} + f(u), \quad t > 0, \quad x \in R^1, \quad y \in G$$

$$\frac{\partial u(t, x, y)}{\partial n} \bigg|_{t > 0, x \in R^1, y \in \partial G} = 0, \quad u(0, x, y) = \chi^-(x). \quad (1.1.3)$$

Here $n = n(y)$ is the outward normal to $\partial G$ at point $y \in \partial G$; $\Delta_{x, y}$ is the Laplacian in $x$ and $y$; $f \in F_1$ as before; $D$ is positive.

As in the one-dimensional case $\alpha^*$ is called the asymptotic speed as $t \to \infty$ if for any $h > 0$

$$\lim_{t \to \infty} \sup_{x > (\alpha^* + h)G \cup \partial G} u(t, x, y) = 0, \quad \lim_{t \to \infty} \inf_{x < (\alpha^* - h)G \cup \partial G} u(t, x, y) = 1.$$

Equation (1.1.3) describes the evolution of particles which diffuse with diffusivity $D$ in the flow having velocity $b$ and take part in the "chemical reaction" governed by the nonlinear term $f(u)$. Of course, one cannot expect that some asymptotic speed will be established if $D$ or $b$ depends on $x$ arbitrarily. Let $D = \text{const}$ and $b$ be independent of $x$. If $b = \text{const}$ then it follows from the results of [15] that the asymptotic speed for problem (1.1.3) is equal to $\alpha^* = b + \sqrt{2cD}$. Now let the velocity of the flow $b$ depend on the point of the cross section of the tube: $b = b(y)$. In the linear case $f(u) \equiv 0$ one can check that

$$\alpha^* = \overline{b} = \frac{1}{|G|} \int_G b(y) \, dy$$

where $|G|$ is the volume of the domain $G$. The last statement is the result of averaging in the $y$-variables: the uniform distribution is the invariant measure for the diffusion governed by $(D/2)\Delta_y$ in $G$ with the normal reflection on the boundary.