EXACT METHODS FOR THE TRANSIENT ANALYSIS OF NONHOMOGENEOUS CONTINUOUS TIME MARKOV CHAINS

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ABSTRACT

Most common performance and reliability models assume that rates associated with events such as arrivals, service completions, failures, repairs, etc. are all constant in time. Many practical systems, however, require time- (age-) dependent rates. For example, the use of Weibull failure rates is quite common in many reliability models. Likewise, most actual local area network (LAN) systems experience surges in the number of users that vary in magnitude over time. These surges may often be approximated by a periodic process. Therefore, nonhomogeneous continuous time Markov chains (CTMCs) may be well suited to model such systems. The transient analysis of time-varying linear systems is highly advanced in the field of systems and control theory. We present a review of some useful results, and then apply them to the analysis of nonhomogeneous CTMCs (especially periodic ones). One of the results of this analysis is, that for a certain class of useful nonhomogeneous CTMCs, a very simple method exists for transforming such a CTMC (and not just a periodic one) to an equivalent homogeneous CTMC that is then amenable to such homogeneous methods as Jensen's method (also known as uniformization or randomization).
1 INTRODUCTION

The transient behavior of a continuous time Markov chain (CTMC) is defined by the well known linear system of first order differential equations:

\[ \dot{\pi}(t) = \pi(t)Q(t) \quad (8.1a) \]

\[ \sum_{i=1}^{m} \pi_i(t) = 1 \quad (8.1b) \]

where \( \pi(t) \) is a row vector, the \( i \)th element of which, \( \pi_i(t) \) \( (i = 1, \ldots, m) \), represents the probability of finding the system in state \( i \) at time \( t \), while \( Q(t) \) \( (\in \mathbb{R}^{m \times m}) \) represents the infinitesimal generator matrix that is constructed from the rates into and out of each state. We assume throughout that \( \pi(t) \) is of finite dimension \( m \). Because the infinitesimal generator is a function of time, eqs.(8.1a) and (8.1b) represent the more general case of a nonhomogeneous CTMC.

If \( Q(t) \) is piecewise continuous (or more weakly, integrable), then there will exist a unique solution to eq.(8.1a) of the form:

\[ \pi(t) = \pi(t_0) \Phi(t, t_0) \quad (8.2) \]

where \( \Phi(t, t_0) \) is the transition probability matrix \( (\in \mathbb{R}^{m \times m}) \) mapping \( \pi(t_0) \) (the initial condition) into \( \pi(t) \). This explicit form exists for any integrable \( Q(t) \).

The general solution for \( \Phi(t, t_0) \) has sometimes been incorrectly given as (see, e.g., [1]):

\[ \Phi(t, t_0) = e^{\int_{t_0}^{t} Q(\tau) \, d\tau} \quad (8.3) \]

where

\[ e^{\int_{t_0}^{t} Q(\tau) \, d\tau} \Delta \sum_{i=0}^{\infty} \frac{1}{i!} \left\{ \int_{t_0}^{t} Q(\tau) \, d\tau \right\}^i. \]

(The matrix exponential is defined by its Taylor series expansion.)

However, it is well known from systems/control theory that Eq.(8.3) is not, in fact, the correct general solution. Eq.(8.3) will only be the solution if \( Q(t) \) and \( \int_{t_0}^{t} Q(\tau) \, d\tau \) commute \( \forall t \) (always true for a scalar \( Q(t) \), but not true in general). Note that if \( Q(t) = f(t)Q \), where \( f(t) \) is an arbitrary scalar function of time and \( Q \) is any valid time-invariant infinitesimal generator matrix, then \( Q(t) \) and its time integral commute. CTMCs of this type arise in reliability analyses of systems with time-dependent failure rates.