QUANTUM MOTION AND ALGEBRAIC
GENERATOR COORDINATE METHOD

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1. CLASSICAL AND QUANTUM MOTION

The symmetries, dynamical symmetries and other noninvariance groups of physical systems play very important role in both classical and quantum physics. It is worthwhile idea to notice that the same group structure is in many cases responsible for description of classical system and its quantum counterpart. This seems to force an idea of a group of motion. The group of motion \( G \) would be responsible for ‘shift’ of the physical system under consideration from one to another physical state.

For classical case the \( G \)-motion (the motion generated by the group \( G \)) of a physical system can be identify with the group elements \( g \in G \). A composition \( g \) of two group elements \( g_1 \) and \( g_2 \), i.e. \( g = g_1 g_2 \), can be interpreted as a composition of two subsequent movements determined by ‘shifts’ \( g_1 \) and \( g_2 \). The inverse element \( g^{-1} \) represents an inverse \( G \)-motion. A composition of \( g \) and \( g^{-1} \) gives the neutral element in \( G \) which corresponds to the motion which does not change the physical state of the system.

For quantum systems we have no unique path of the motion but the system can choose different paths with some probability amplitudes which implies that quantum systems should be described by the formal integrals over the group manifold \( G \) of the following form

\[
\int_G dg u(g) g,
\]

where \( u \in L^1(G) \) describe amplitudes of the \( G \)-motion (\( G \)-amplitudes) and \( dg \) denotes the left invariant Haar measure. A classical motion corresponds to the amplitude of motion \( u \) with non-zero values strongly concentrated around a given point \( g_0 \) (for a given moment of time) to have

\[
\int_G dg u(g) g \sim g_0.
\]
The composition of two motions (1)

\[ \int_G dgu_1(g)g \int_G d'g' u_2(g')g' = \int_G d(g(u_1 \ast u_2))(g)g, \]  

(3)

where

\[ (u_1 \ast u_2)(g) = \int_G d'g' u_1(g')u_2(g'^{-1}g) \]  

(4)

leads to the element of the form (1) but with \(G\)-amplitude of \(G\)-motion given by the convolution (4) of the partial \(G\)-amplitudes \(u_1\) and \(u_2\).

An analogue of the inverse motion one can get defining involution of the element (1) as follows

\[ \left( \int_G dgu_1(g)g \right)^\dagger = \int_G du^*(g)g^{-1} \rightarrow g_0^{-1}. \]  

(5)

In addition, because of existence of quantum interference effect one can define a sum (interference) of motions by the equation

\[ \int_G dgu_1(g)g + \int_G dgu_2(g)g \equiv \int_G d(g(u_1 + u_2))(g)g. \]  

(6)

This way we came to the algebra of quantum motions generated by the group of motion \(G\). This algebra can be identify with the convolutive group algebra of complex functions (\(G\)-amplitudes) belonging to \(L^1(G, dg)\) with involution defined by [1]

\[ u^\dagger(g) \equiv \Delta_G(g)u^*(g^{-1}), \]  

(7)

where \(\Delta_G\) denotes the modular function of the group \(G\). We define the modular function by the relation

\[ d(g') \equiv \Delta_G(g^{-1})dg'. \]  

(8)

2. THE METASTATE

In the first section we have derived the Banach algebra with involution

\[ \mathcal{R} \equiv (L^1(G), *, d) \]  

(*)

which can describe quantum motion. The group of motion determines what kind of motion is under consideration. The question arise what is the physical system moving? This is determined by the metastate known from algebraic approach to quantum mechanics [2-4]. The metastate is a non-negative, appropriately normalized continous linear functional on the algebra of motions \(\mathcal{R}\). In our case of \(L^1(G)\) type algebra the general form of the metastate is [4,5]

\[ (\rho; u) = \int_G dgu(g) \rho(g), \]  

(9)

where the complex function \((\rho; g)\) called the metastate kernel (m.k.) satisfies the following conditions:

(10a) \(\rho(g) : G \rightarrow C\)

(10b) \(\rho(e) = 1\), where \(e\) denotes the unity in \(G\)