SYMMETRY BREAKING IN THE
PERIOD DOUBLING ROUTE TO CHAOS

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1. INTRODUCTION

The period doubling route to chaos is not quite as universal as one is often made to believe. In many practical situations the well known sequence \(1 \rightarrow 2 \rightarrow 4 \rightarrow 8 \ldots \rightarrow \text{chaos} \) is interrupted or even completely broken off before chaos is reached [1-6]. The present paper is about one particular (rather mild) kind of interruption: a symmetry breaking bifurcation, already at the level of period 2. The period 2 solution, instead of period doubling to a period 4 solution, bifurcates into two non-symmetric solutions of period 2. Subsequently both of the two newly formed period 2 solutions resume the period doubling route as if nothing had happened. The complete scenario can thus be described by the following sequence: \(1 \rightarrow 2 \rightarrow 2 \times (2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow \ldots \rightarrow \text{chaos})\).

As early as 1979 this sequence was found by R.M. May to be the typical route to chaos for a one-dimensional map with two critical points [7,8], but at the time the relevance to practical problems seemed to be marginal and even May himself thought that it was interesting mainly for its "mathematical intricacies". However, there is more to it than that. The main point of the present paper is that the symmetry breaking can be seen in real life systems and therefore the emphasis will lie on two examples: in section 2 we describe a ball bouncing on a vibrating table, and in section 3 a pendulum which is being driven up and down at its point of support. In doing so it will become clear that the symmetry breaking interruption is not restricted to one-dimensional systems. In section 4 we extend May’s analysis to two dimensions and come up with a model map which accurately represents all of the previously observed behaviour.

2. FIRST EXAMPLE: THE BOUNCING BALL

Consider a small ball, of negligible mass, which is bouncing up and down on a vibrating table. This system, known as the Fermi-Ulam model, has been studied by a number of people, both theoretically [9,10,11] and experimentally [12-20]. The (vertical) motion of the table is given by \(z(t) = z_0 \cos(wt)\), in which the amplitude \(z_0\) is taken to be much smaller than the height of the jumps of the ball. The situation is sketched in figure 1.
Let \( v_1 \) be the (upward) velocity of the ball just after the \( i \)-th bounce, which takes place at time \( t_i \). Neglecting air resistance and neglecting any variations in the height of the table, it takes the ball a time \( 2v_1/g \) to make its jump and return to the table. So we have:

\[
 t_{i+1} = t_i + \frac{2}{g} v_1 .
\]  

Furthermore, assuming that the bounce is a perfectly elastic one, the velocity of the ball is augmented by twice the velocity of the table. That is,

\[
 v_{i+1} = v_i - 2z_0 \omega \sin \omega t_{i+1} .
\]  

The above two equations fully describe the dynamics of the system, in the absence of dissipation. Eliminating the velocity via \( v_i = g (t_{i+1} - t_i)/2 \), the two equations can be combined into one, relating the times \( t_i \), \( t_{i+1} \) and \( t_{i+2} \) of successive bounces:

\[
 t_{i+2} = 2t_{i+1} - \frac{4z_0 \omega}{g} \sin \omega t_{i+1} - t_i .
\]  

Equivalently, with \( x_i = \omega t_i \) and \( y_i = \omega t_{i-1} \) (both modulo \( \omega T(\omega) = 2\pi \), where \( T(\omega) \) is the period of the table) this can be written in the form of a two-dimensional area preserving map:

\[
 \begin{align*}
 x_{i+1} &= 2x_i - A \sin x_i - y_i , \\
 y_{i+1} &= x_i .
 \end{align*}
\]  

Here

\[
 A = \frac{4z_0 \omega^2}{g}
\]  

is the so-called chaos parameter \( (A \geq 0) \). If we increase the value of \( A \), by increasing the value of \( \omega \) (not the value of \( z_0 \), since this must remain small) we witness a transition from order to chaos via period doubling bifurcations.