8.1. Introduction

The prefix "m" stands for "many" ("M" stands for "Minkowski"), and the purpose of this chapter is to demonstrate that M-matrices are very useful. We shall describe, or at least mention, some of their applications to ecology, numerical analysis, probability, mathematical programming, game theory, control theory, and matrix theory.

Our notation is standard. For the convenience of the reader, it will be summarized at the end of the chapter.

8.2. M-matrices: Definitions and Properties

An \( M \)-matrix \( A \in \mathbb{R}^{n \times n} \) is a matrix of the form \( A = \alpha I - B \), where \( B \geq 0 \) (\( B \) is elementwise nonnegative) and \( \alpha \geq \rho(B) \). (By the Perron–Frobenius theorem e.g., [3], p. 26, \( \rho(B) \), the spectral radius of \( B \), is an eigenvalue of \( B \).)

A matrix of the form \( \alpha I - B \), \( B \geq 0 \) is called a \( Z \)-matrix. Observe that
a Z-matrix \( A \) is an M-matrix if and only if \( A + \epsilon I \) is nonsingular for all \( \epsilon > 0 \). (If \( A = \alpha I - B, B \geq 0, \alpha < \rho(B) \), then \( A + (\rho(B) - \alpha)I \) is singular.)

The following theorem collects conditions that characterize nonsingular M-matrices.

**Theorem 8.1.** Let \( A = \alpha I - B, B \geq 0 \). Then the following statements are equivalent:

a. \( \alpha > \rho(B) \),

b. \( A \) is positive stable: If \( \lambda \) is an eigenvalue of \( A \), then \( \Re \lambda > 0 \),

c. \( A \) is nonsingular and \( A^{-1} \geq 0 \),

d. \( Ax \) is positive for some positive vector \( x \),

e. The principal minors of \( A \) are positive,

f. The leading principal minors of \( A \) are positive.

Conditions (b), (c), and (e) are due to Ostrowski [22], who introduced the concept of M-matrices. Condition (e) is known in the economics literature as the Hawkins–Simon condition [13]. Condition (d) is due to Schneider [26], and Fan [8], and condition (f) to Fiedler and Ptak [10]. Many additional characterizations of nonsingular (and of singular) M-matrices are given in [3], chapter 6.

A subset of the set of all M-matrices that contains the nonsingular M-matrices and whose matrices share many of their properties is the set of group-invertible M-matrices (M-matrices with "property c"). To define this set, we need the concept of semiconvergence.

A matrix \( T \) is semiconvergent if \( \lim_{k \to \infty} T^k \) exists. \( T \) is convergent if this limit is the zero matrix. \( T \) is convergent if and only if \( \rho(T) < 1 \) and semiconvergent if and only if \( \rho(T) \leq 1 \); if \( \rho(T) = 1 \), then 1 is the only eigenvalue of \( T \) on the unit circle, and its elementary divisors are linear.

A matrix \( A \) is a group-invertible M-matrix if it can be written as \( A = \alpha I - B, B \geq 0 \) where \( \alpha^{-1}B \) is semiconvergent.

To clarify the terminology, recall (e.g., [1]) that \( X \) is the group inverse of \( A, X = A^g \), if \( AXA = A, XAX = X, \) and \( AX =XA \). It exists (and is unique) if and only if rank \( A = \text{rank } A^2 \), or equivalently if \( A \) is nonsingular or the elementary divisors of 0 are linear. It follows that an M-matrix \( A \) is group invertible if and only if \( A^g \) exists. (To prove "if", suppose that \( \alpha^{-1}B \) has an eigenvalue different from 1 on the unit circle. Then, for \( \epsilon > 0 \), \((\alpha + \epsilon)^{-1}(\epsilon I + B)\) will have only 1 as an eigenvalue on the unit circle. The "only if" part is clear.)