WEAK DIFFERENTIATION AND GRADIENT ESTIMATION FOR DISCRETE EVENT DRIVEN PROCESSES

B. Heidergott

EURANDOM, P.O.Box 513, 5600 MB Eindhoven, the Netherlands, This research is supported by Deutsche Forschungsgemeinschaft under grant He 3139/1-1.

heidergott@eurandom.tue.nl

F. Vázquez-Abad

DIRO, Université de Montréal, C.P. 6128 Succursale Centre-Ville, H3C 3J7 Canada.

Supported by FCAR and NSERC grants; partly completed while the author was on leave at the DEEE, University of Melbourne, Australia.

vazquez@IRO.UMontreal.CA

Keywords: Gradient Estimation, Weak Derivatives, Score Function, Smoothed Perturbation Analysis

Abstract

We provide a unified framework for gradient estimation for discrete event systems (DES). We establish a product rule of differentiation for DES. Applying this product rule to known gradient estimation techniques like the Score Function, Smoothed Perturbation Analysis and Weak Derivatives, we prove unbiasedness of the resulting estimators for finite horizon experiments.

Introduction

The last two decades have witnessed a great interest in the area of gradient estimation [4, 6, 2, 3, 9]. The methods available are legion and even experts find it difficult to oversee the various methods and estimators. It is widely accepted today that to estimate the derivative of a continuous performance measure the Infinitesimal Perturbation Analysis (IPA) method is usually the preferred method. Hence, today the mathematical discussion is about what method to use in the non-IPA cases, such like inventory models.

R. Boel et al. (eds.), Discrete Event Systems
Consider the following gradient estimation problem for finite horizon performance indices of discrete event systems (DES). Let \( \{X_\theta(k)\} \) denote the general state-space Markov Chains describing the system process, often called Generalised Semi-Markov Processes, see [4] or [6]. The pathwise analysis states the problem in terms of estimating

\[
F = \frac{d}{d\theta} \int g(X_\theta(1, \omega), \ldots, X_\theta(n, \omega)) \, \mathbb{P}(d\omega)
\]

where \( g \in \mathbb{R} \) is a sample performance and \( \{X_\theta(k)\} \) represents the system process on a measurable space \((\Omega, \mathbb{P})\) which is independent of \( \theta \). The control parameter is in a compact set \( \theta \in \Theta \subset \mathbb{R} \) and it is assumed that the above derivative is well defined on \( \Theta \). Small perturbations of \( \theta \) are studied for each trajectory \( \omega \) to evaluate the above sensitivity. If \( h_\theta = g \circ (X_\theta(1), \ldots, X_\theta(n)) \) is a.s. uniformly Lipschitz continuous in a neighborhood of \( \theta \), then the Dominated Convergence Theorem ensures that the derivative and expectation can be interchanged and the IPA estimator \( \frac{d}{d\theta} h_\theta(\omega) \) is unbiased. In the presence of discontinuities, the Smoothed Perturbation Analysis (SPA) of [5, 2] can be applied here conditioning on the trajectories where jumps occur for small perturbations of size \( \Delta \). Replacing \( h_\theta \) by its conditional expectation, where the discontinuities have been integrated, results in a Lipschitz continuous function for which IPA can sometimes be applied.

The dual view to the sample path analysis is called the "weak differentiation" approach, since it uses weak topology concepts. Under this alternative model, the underlying probability measure contains all the dependency in \( \theta \), so that:

\[
F = \frac{d}{d\theta} \int h_\theta(\omega) \, \mathbb{P}(d\omega) = \int g(x) \, \frac{d}{d\theta} P_\theta(dx)
\]

where \( P_\theta \) is the induced probability distribution of \((X_\theta(1), \ldots, X_\theta(n))\).

In this paper a weak differentiation approach is stated, which will establish the basis for a methodology that recovers the Likelihood Ratio method of [8, 9], the Weak Derivatives of [7] and Discontinuous Perturbation Analysis, particularly the Smoothed Perturbation Analysis (SPA) for threshold parameters (see [2]), and the Rare Perturbation Analysis (RPA) of [1]. The weak differentiation formulation allows us to study the general conditions under which the estimators are unbiased, or consistent, for a class of performance functions. Our analysis is restricted here to the class of bounded performance measures. However, the results can be extended for uniformly integrable performances and we are currently studying the general integrability conditions.