1. PRELUDE

Continuous quantum phase transitions have attracted much attention in this decade both from experimentalists as well as from theorists. (For reviews see Refs.1–4). These transitions, taking place at the absolute zero of temperature, are dominated by quantum and not by thermal fluctuations as is the case in classical finite-temperature phase transitions. Whereas time plays no role in a classical phase transition, being an equilibrium phenomenon, it becomes important in quantum phase transitions. The dynamics is characterized by an additional critical exponent, the so-called dynamic exponent, which measures the asymmetry between the time and space dimensions. The natural language to describe these transitions is quantum field theory. In particular, the functional-integral approach, which can also be employed to describe classical phase transitions, turns out to be highly convenient.

The subject is at the border of condensed matter and statistical physics. Typical systems being studied are superfluid and superconducting films, quantum-Hall and related two-dimensional electron systems, as well as quantum spin systems. Despite the diversity in physical content, the quantum critical behavior of these systems shows surprising similarities. It is fair to say that the present theoretical understanding of most of the experimental results is scant.

The purpose of this Lecture is to provide the reader with a framework for studying quantum phase transitions. A central role is played by a repulsively interacting Bose gas at the absolute zero of temperature. The universality class defined by this paradigm is believed to be of relevance to most of the systems studied. Without impurities and a Coulomb interaction, the quantum critical behavior of this system turns out to be surprisingly simple. However, these two ingredients are essential and have to be included. Very general hyperscaling arguments are powerful enough to determine the exact value of the dynamic exponent in the presence of impurities and a Coulomb interaction, but the other critical exponents become highly intractable.

The emphasis in this Lecture will be on effective theories, giving a description of the system under study valid at low energy and small momentum. The rationale for this is the observation that the (quantum) critical behavior of continuous phase transitions is determined by such general features as the dimensionality of space, the
symmetries involved, and the dimensionality of the order parameter. It does not depend on the details of the underlying microscopic theory. In the process of deriving an effective theory starting from some microscopic model, irrelevant degrees of freedom are integrated out and only those relevant for the description of the phase transition are retained. Similarities in the critical behavior in different systems can, accordingly, be more easily understood from the perspective of effective field theories.

The ones discussed in this Lecture are so-called phase-only theories. They are the dynamical analogs of the familiar O(2) nonlinear sigma model of classical statistical physics. As in that model, the focus will be on phase fluctuations of the order parameter. The inclusion of fluctuations in the modulus of the order parameter is generally believed not to change the critical behavior. Indeed, there are convincing arguments that both the Landau-Ginzburg model with varying modulus and the O(n) nonlinear sigma model with a fixed modulus belong to the same universality class. For technical reasons a direct comparison is not possible, the Landau-Ginzburg model usually being investigated in an expansion around four dimensions, and the nonlinear sigma model in one around two.

In the case of a repulsively interacting Bose gas at the absolute zero of temperature, the situation is particularly simple as phase fluctuations are the only type of field fluctuations present.

This Lecture covers exclusively lower-dimensional systems. The reason is that in three space dimensions and higher the quantum critical behavior is in general Gaussian and, therefore, not very interesting.

Since time, and how it compares to the space dimensions is an important aspect of quantum phase transitions, Galilei invariance will play an important role in the discussion.

1.1 Notation

We adopt Feynman’s notation and denote a spacetime point by $x = x_\mu = (t, \mathbf{x})$, $\mu = 0, 1, \cdots, d$, with $d$ the number of space dimensions, while the energy $k_0$ and momentum $\mathbf{k}$ of a particle will be denoted by $k = k_\mu = (k_0, \mathbf{k})$. The time derivative $\partial_0 = \partial / \partial t$ and the gradient $\nabla$ are sometimes combined in a single vector $\hat{\partial}_\mu = (\partial_0, -\nabla)$. The tilde on $\hat{\partial}_\mu$ is to alert the reader for the minus sign appearing in the spatial components of this vector. We define the scalar product $k \cdot x = k_\mu x_\mu = k_0 t - \mathbf{k} \cdot \mathbf{x}$ and use Einstein's summation convention. Because of the minus sign in the definition of the vector $\hat{\partial}_\mu$ it follows that $\hat{\partial}_\mu a_\mu = \partial_0 a_0 + \nabla \cdot \mathbf{a}$, with $a_\mu$ an arbitrary vector.

Integrals over spacetime are denoted by

$$\int = \int_{t, \mathbf{x}} dt \, d^d x,$$

while those over energy and momentum by

$$\int = \int_{k_0, \mathbf{k}} \frac{dk_0 \, d^d k}{2\pi (2\pi)^d}.$$

When no integration limits are indicated, the integrals are assumed to run over all possible values of the integration variables.

Natural units $\hbar = c = k_B = 1$ are adopted throughout.