10. UNSCALED EMPIRICAL LORENZ PROCESSES.

Let us recall that we assume the continuity of \( F \), and consider the integral

\[
U_n(y) = \int_0^y Q(x^-) dE_n(x).
\]

If \((k-1)/n \leq y < k/n\) for some \( k=1, \ldots, n \), then, apart from the set where either ties occur among \( X_1, \ldots, X_n \) or they fall into discontinuity points of \( Q(\cdot) \), we have

\[
\frac{1}{n} \sum_{i=1}^{[ny]+1} X_i:n.
\]

and the same argument shows that the value of our integral at \( y=1 \) is \( \bar{X}_n \) almost surely. Thus we arrive at our basic observation for the present section:

\[
\text{(10.1) } \mathbb{P} \left\{ \sup_{0 < y \leq 1} \left| G_n(y) - \int_0^1 Q(x^-) dE_n(x) \right| = 0 \right\} = 1
\]

for each \( n \), where

\[
G_n(y) = \bar{X}_n L_n(y) = \begin{cases} \frac{1}{n} \sum_{i=1}^{[ny]+1} X_i:n', & 0 < y < 1, \\ \bar{X}_n, & y = 1 \end{cases}
\]

is the unscaled empirical Lorenz curve. Introducing the unscaled Lorenz curve

\[
G_F(y) = uL_F(y), \quad 0 \leq y \leq 1,
\]

where \( L_n \) and \( L_F \) are as in (1.10) and (1.8), respectively, we have the following consistency result.

**THEOREM 10.1.** If \( \mu = \int_0^1 Q(y) dy < \infty \), then

\[
\Delta_n^{(7)} = \sup_{0 < y < 1} |G_n(y) - G_F(y)| \overset{a.s.}{\rightarrow} 0.
\]

**Proof.** Let \( \varepsilon > 0 \) be arbitrarily small and choose \( \beta \in (0,1) \) so large that

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(10.2) \[ I_1(\beta) = \int_{1-\beta}^{1} Q(x) \, dx < \epsilon/2. \]

The latter is possible by the proof of Lemma 1 on p. 148 of Feller (1966). Now by (10.1)

\[
\Delta_n^{(7)} \overset{a.s.}{=} \sup_{0 < y < 1} \left| \int_{0}^{y} Q(x-) dE_n(x) - \int_{0}^{y} Q(x-) \, dx \right| \\
\leq \sup_{0 < y < 1} \left| \int_{0}^{y} Q(x-) dE_n(x) - \int_{0}^{y} Q(x-) \, dx \right| \\
+ \sup_{0 < y < 1} \left| G_F(U_n(y)) - G_F(y) \right|,
\]

where the second term goes to zero almost surely by the continuity of \( G_F \) and the first term is not greater than

\[
\sup_{0 < y < 1-\beta} \left| \int_{0}^{y} Q(x-) dE_n(x) - \int_{0}^{y} Q(x-) \, dx \right| + \sup_{1-\beta < y < 1} \left| \int_{0}^{y} Q(x-) dE_n(x) - \int_{0}^{y} Q(x-) \, dx \right| \\
\leq 2 \sup_{0 < y < 1-\beta} \left| \int_{0}^{y} Q(x-) dE_n(x) - \int_{0}^{y} Q(x-) \, dx \right| + I_1(\beta) + \int_{1-\beta}^{1} Q(x) \, dx \\
\leq 2 \Delta_n^{(1)} + 2Q(1-\beta) \sup_{0 < y < 1} \left| E_n(y) - y \right| \\
+ I_1(\beta) + \frac{1}{n} \sum_{i=1}^{n} Q(U_i) \chi(\{U_i > 1-\beta\}),
\]

upon integrating by parts in the last step, where \( \Delta_n^{(1)} \) is that of Lemma 3.1. By this lemma, by Glivenko-Cantelli and by the strong law of large numbers we have \( \limsup_{n \to \infty} \Delta_n^{(7)} < 2I_1(\beta) \leq \epsilon \) a.s. for all \( \epsilon > 0 \) and hence the theorem is proved.

**THEOREM 10.2.** If \( Q = F^{-1} \) is continuous on \([0,1]\) and \( \text{EX}^2 < \infty \), then

\[
\Delta_n^{(8)} = \sup_{0 < u < 1} \left| g_n(u) - \Gamma_n(u) \right| \overset{P}{\to} 0
\]

where

\[
g_n(u) = n^b (G_n(u) - G_F(u)), \quad 0 \leq u \leq 1,
\]

and the sequence \( \Gamma_n \) of zero-mean Gaussian processes is defined as

\[
(10.3) \quad \Gamma_n(u) = \int_{0}^{u} B_n(y) \, dQ(y) \\
= Q(u)B_n(u) - \int_{0}^{u} Q(y) \, dB_n(y),
\]