12. DISCUSSION OF RESULTS ON EMPIRICAL LORENZ PROCESSES

1) The limit process of the unscaled Lorenz process \( q_n(u) = n^{-1}(G_n(u) - G_F(u)) \), \( 0 \leq u \leq 1 \), is the mean-zero Gaussian process

\[
\Gamma(u) = \Gamma_F(u) = \int_0^u B(y) dQ(y), \quad 0 \leq u \leq 1.
\]

If \( F_{\theta,\sigma}(x) = F((x-\theta)/\sigma) \), \( -\infty < \theta < \infty \), \( \sigma > 0 \), then

\[
\Gamma_{F_{\theta,\sigma}}(u) = \sigma \Gamma_F(u),
\]

that is, \( \Gamma_F \) is shift-free. The covariance function of \( \Gamma_F \) is

\[
\sigma_3(s, t) = E\Gamma_F(s)\Gamma_F(t) = \int_0^s \int_0^t (\min(u, v) - uv) dQ(u) dQ(v)
\]

for \( 0 \leq s, t \leq 1 \). If \( 0 \leq s \leq t \leq 1 \) we obtain

\[
\sigma_3(s, t) = 2 \int_0^s (1-u)\{Q(u) - N^1_F(u)\} dQ(u)
\]

\[
+ \{N^1_F(t) - N^1_F(s)\}\{Q(s) - N^1_F(s)\},
\]

where

\[
N^1_F(u) = H^{-1}_F(u) - t_F = \int_0^u (1-y) dQ(y),
\]

and hence the variance function is

\[
\sigma_3^2(t) = 2 \int_0^t (1-u)\{Q(u) - N^1_F(u)\} dQ(u).
\]

2) The limit process of the empirical Lorenz process \( \lambda_n(u) = n^{-1}(L_n(u) - L_F(u)) \), \( 0 \leq u \leq 1 \), is the mean-zero Gaussian process

\[
\Lambda(u) = \Lambda_F(u) = \frac{1}{u} \left\{ \int B(y) dQ(y) - \frac{1}{0} \int B(y) dQ(y) \right\},
\]

\( 0 \leq u \leq 1 \). If \( F_{\sigma}(x) = F_{0, \sigma}(x) = F(x/\sigma) \), \( \sigma > 0 \), then

\[
\Lambda_{F_{\sigma}}(u) = \Lambda_F(u), \quad 0 \leq u \leq 1,
\]

i.e., \( \Lambda_F \) is scale-free. The covariance function of \( \Lambda_F \) is

\[
\sigma_4(s, t) = E\Lambda_F(s)\Lambda_F(t)
\]

\[
= u^{-2}\{\sigma_3(s, t) + L_F(s)L_F(t)\sigma_3(1, 1)
\]

\[
- L_F(t)\sigma_3(s, 1) - L_F(s)\sigma_3(t, 1)\}.
\]

Although Goldie's (1977) representation of the limit process \( \Lambda_F \)
is the same as ours, he gives the covariance function in a more complicated form in terms of the truncated second-moment function

\[ Q(t) \int_0 x^2 dF(x) \]

arising in a natural way in his proof of the tightness of the sequence \( \{ l_n(\cdot) \} \).

The variance function is

\[ \sigma_4^2(t) = E \Lambda_F^2(t) \]

\[ = \frac{1}{\mu^2} \left\{ 2 \int_0^t (1-u) \{ Q(u)-N^1_F(u) \} \, dQ(u) \right\} \left[ 1-2L_F(t) \right] \]

\[ + 2L_F^2(t) \int_0^1 (1-u) \{ Q(u)-N^1_F(u) \} \, dQ(u) \]

\[ -2L_F(t) [N^1_F(1)-N^1_F(t)] [Q(t)-N^1_F(t)] \} . \]

3) Since we could not identify \( \Gamma_F \) or \( \Lambda_F \) as a known process for any specified \( F \) and could not compute the distribution of any of their functionals, we must use the bootstrap method of Section 17. We can, of course, again draw consequences of the convergence theory of Lorenze processes pointwise. Let us choose a fixed point \( u \in (0,1) \) and consider the estimator \( \sigma_4^2(u) \) of the limiting variance \( \sigma_4^2(u) \) obtained by replacing \( \mu \) by the sample mean \( \overline{X}_n \), \( L_F \) by the empirical Lorenz curve \( L_n \), \( Q \) by the sample quantile function \( Q_n \) in (8.5) and \( N^1_F \) by its empirical counterpart

\[ N^1_F(u) = \int_0^u (1-y) dQ_n(y) \]

in the definition of \( \sigma_4^2(u) \) above. Note that \( N^1_n(l) = \overline{X}_n - \chi_{1:n} \)

**THEOREM 12.1.**

(i) **If** \( \mu < \infty \) **then**

\[ L_n(u) \overset{\text{a.s.}}{\longrightarrow} L_F(u) . \]

(ii) **If** \( Q \) **is continuous at** \( u \) **and** \( EX^2 < \infty \), **then**

\[ \lim \frac{L_n(u)}{\sigma_4^2(u)} \leq x = \Phi(x), \quad -\infty < x < \infty . \]

(iii) **If** \( EX^2 < \infty \) **and** \( Q \) **is continuous on** \( [0, u+\varepsilon) \) **with any small** \( \varepsilon > 0 \), **then**

\[ \lim_{n \to \infty} \text{pr} \left\{ L_n(u) - \frac{\sigma_4^2(u)}{\sqrt{n}} < L_F(u) < L_n(u) + \frac{\sigma_4^2(u)}{\sqrt{n}} \right\} = 2\Phi(x) - 1 . \]