Intrinsic Riemannian geometry of surfaces

7

7.1 Parallel translation and connections

Definition. Let \( M \) be an oriented Riemannian manifold of dimension 2. Let \( T(M) \) denote the tangent bundle of \( M \). Let

\[
S(M) = \{(m, v) \in T(M); \langle v, v \rangle = 1 \}.
\]

\( S(M) \) is called the sphere bundle, or circle bundle, of \( M \).

The notation \((m, v)\) for a point of \( T(M) \) \{or \( S(M) \)\} is redundant since \( v \in T(M, m) \). Nevertheless, we use it to emphasize that \( v \) is a tangent vector at \( m \).

Remarks

1. \( S(M) \) is a smooth manifold of dimension 3. The function \( f: T(M) \to \mathbb{R}^3 \), given by \( f(m, v) = \langle v, v \rangle \), is smooth, and \( df \neq 0 \) whenever \( f = 1 \), so the implicit function theorem applies.

2. Note that the circle \( S^1 = \{z \in \mathbb{C}; |z| = 1 \} \) is a group under (complex) multiplication. Since \( e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} \), the group \( S^1 \) is just the group of rotations of the oriented plane \( \mathbb{R}^2 \). This group acts on \( S(M) \): there exists a smooth map

\[
A: S^1 \times S(M) \to S(M)
\]

given by

\[
A(g, (m, v)) = (m, gv) \quad (g \in S^1; (m, v) \in S(M)),
\]

where \( gv \) is the image of the vector \( v \) under rotation by \( g \) in the oriented plane \( T(M, m) \) (Figure 7.1). So, if \( g = e^{i\theta} \), and \( \{v_1, v_2\} \) is any oriented orthonormal basis for \( T(M, m) \), then \( v = c_1v_1 + c_2v_2 \) for some \( c_1, c_2 \in \mathbb{R} \), and

\[
fv = (c_1 \cos \theta - c_2 \sin \theta)v_1 + (c_2 \sin \theta + c_2 \cos \theta)v_2.
\]
We shall often denote $A(g, (m, v))$ by $g(m, v)$. Then $g: S(M) \to S(M)$ is a smooth map for each $g \in S^1$.

\[ g \colon S(M) \to S(M) \]

Figure 7.1

(3) If $\pi: S(M) \to M$ denotes projection, then $\pi^{-1}(m)$ is just the unit circle in $T(M, m)$. Moreover, if $(m, v_1)$ and $(m, v_2)$ are any two elements of $\pi^{-1}(m)$, then there exists a unique $g \in S^1$ such that $(m, v_2) = g(m, v_1)$. (Take $g = e^{i\theta}$, where $\theta$ is the positive angle of rotation from $v_1$ to $v_2$.)

(4) $S(M)$ is locally a product space. For let $U$ be a coordinate neighborhood in $M$, with coordinate functions $(x_1, x_2)$. Let $e_1$ be the vector field $(\partial/\partial x_1)/\|(\partial/\partial x_1)\|$, where $\|(\partial/\partial x_1)\| = \langle(\partial/\partial x_1), (\partial/\partial x_1)\rangle^{1/2}$. Then $e_1$ is a smooth vector field on $U$, which is everywhere of length 1. Thus $e_1$ defines a smooth map

\[ c : U \to \pi^{-1}(U) \quad \text{by} \quad c(m) = (m, e_1(m)). \]

Clearly $\pi \circ c = i_U$. Now define $B : U \times S^1 \to \pi^{-1}(U)$ by

\[ B(m, g) = gc(m) = (m, ge_1(m)) = A(g, (m, e_1(m))). \]

Then it is easy to verify that $B$ is smooth, injective, and surjective; and that $dB$ is everywhere nonsingular so that $B^{-1}$ is also smooth.

(5) It is not true that $S(M)$ is globally a product of $S^1$ with $M$. If there exists a smooth nonzero vector field on $M$, then the above argument shows that $S(M)$ is diffeomorphic with $M \times S^1$. However, there do not exist such nonzero vector fields in general. (For example, $M = S^2$.)

For $M = \mathbb{R}^2$, the notion of translating a tangent vector parallel to itself is clear. We now propose to generalize it and introduce the concept of parallel translation of tangent vectors on arbitrary 2-dimensional oriented Riemannian manifolds. It will turn out that we will be able to parallel translate vectors along curves from one point to another, but that the result will depend on the curve. In particular, if we parallel translate around a closed curve, we may not get back to our original vector. The new vector will differ from the original vector by a rotation; i.e., by an element of $S^1$. For $M = \mathbb{R}^2$, a "flat" space, this rotation is zero. For arbitrary $M$, this rotation (or, more precisely, the limit of it as the curve shrinks to a point $m$) will measure the "curvature" of $M$ at $m$.

We develop this notion of parallel translation in order to obtain an intrinsic meaning for curvature of the Riemannian manifold $M$; that is, a meaning independent of any ambient space in which $M$ may lie. In Chapter 8 we will interpret this curvature differently when $M$ is a submanifold of $\mathbb{R}^3$. 

176