Two Examples of Parameter Estimation for Stochastic Partial Differential Equations

M. HÜBNER, R. KHASMINSKII, B.L. ROZOVSKII

Abstract

We study parameter estimation for two types of parabolic stochastic PDE’s. Examples considered in this article suggest that asymptotic properties of maximum likelihood estimators (MLE’s) for Galerkin approximations to these SPDE’s depend critically on certain properties of the distributions of solutions to the original equations. In particular, singularity of the distributions for different values of the parameter provides for consistency of the MLE as the dimension of the approximation approaches infinity.

1 Introduction

The aim of this work is to consider some interesting phenomena arising in parameter estimation problems for stochastic partial differential equations (SPDE’s). We consider the Dirichlet problem with zero boundary conditions for the equation

\[ du_\epsilon(t, x) = \mathcal{L}_\theta u_\epsilon(t, x) dt + \epsilon dW_Q(t, x) \]

where \( W_Q(t, x) \) is a Wiener process in \( L_2(0, 1) \) with the nuclear covariance operator \( Q \). This problem is studied in two cases:

(1.1) \[ \mathcal{L}_\theta u_\epsilon(t, x) = \frac{\partial^2}{\partial x^2} u_\epsilon(t, x) + \theta u_\epsilon(t, x) \]

and

(1.2) \[ \mathcal{L}_\theta u_\epsilon(t, x) = \theta \frac{\partial^2}{\partial x^2} u_\epsilon(t, x) . \]

For \( \theta \in R_+ \), both equations belong to the same class and their fundamental analytical properties (existence, uniqueness, smoothness of solutions, asymptotic properties, etc.) are identical. However, as we demonstrate below, when it comes to parameter estimation, these two cases are very

---

1 Work partially supported by NSF Grant No. DMS-9002997 and ONR Grant No. N00014-91-J-1526
different. The first example is in some sense routine, because the measures $P_\theta^\epsilon$ generated by the solutions of these equations become singular only if the intensity parameter $\epsilon$ for the white Gaussian noise is equal to 0. The results for this case are analogous to results obtained by [Ibragimov, Khasminskii (1981)] and [Kutoyants (1984)] where this problem is studied for ordinary SDEs. Here it is possible to consider a suitable $N$-dimensional projection of the observation and to prove that the solution of this finite dimensional estimation problem converges uniformly in $\epsilon$ to the solution of the infinite dimensional problem. The family of measures have the LAN property and the MLE $\hat{\theta}_{N,\epsilon}$ is asymptotically normal.

A completely different situation occurs in the second example. Here even for any positive $\epsilon$ the measures $P_\theta^\epsilon$ are singular for different $\theta$'s, although the measures corresponding to the finite dimensional projections are absolutely continuous. The LAN property and asymptotic normality of the MLE as $\epsilon \to 0$ for these projections hold here, too, but not uniformly in $N$. We will show that the accuracy of the MLE will grow very quickly in $N$ even for fixed $\epsilon = \epsilon_0$ and the problem is to study properties of best estimators for $N \to \infty$. The MLE $\hat{\theta}_{N,\epsilon_0}$ is asymptotically normal and asymptotically efficient for any bounded loss function.

2 The Case of Absolutely Continuous Distributions

Let us fix a probability space $(\Omega, \mathcal{F}, P)$ and consider the process $u^\epsilon(t, x)$ $0 < x < 1$, $0 \leq t \leq T$ governed by

$$du^\epsilon(t, x) = (\Delta u^\epsilon(t, x) + \theta u^\epsilon(t, x)) \, dt + \epsilon dW_Q(t, x)$$

where $\Delta = \frac{\partial^2}{\partial x^2}$. We assume that $\epsilon$ is a small parameter, $\epsilon \to 0$ and $\theta \in \Theta \subset \mathbb{R}$.

Equation (2.1) is considered together with initial and boundary conditions

$$u^\epsilon(0, x) = f(x), \quad f \in L_2(0, 1)$$

(2.2)

$$u^\epsilon(t, 0) = u^\epsilon(t, 1) = 0, \quad 0 \leq t \leq T$$

$Q$ is the covariance operator for the Wiener process $W_Q(t, x)$, so that

$$W_Q(t, x) = Q^{1/2}W(t, x)$$

where $W(t, x)$ is a cylindrical Brownian motion in $L_2(0, 1)$ (a Wiener process in $L_2(0, 1)$ with unity covariance operator). It is a standard fact (see