COMPUTER ALGEBRA IN SPINOR CALCULATIONS

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Abstract. In this paper, some relationships between spinor and vector fields are investigated in detail, with the aid of computer algebra (calculations have been performed with MAPLE). In particular, an exact paradigm based on the representations of vector spaces is shown, strictly linking Pauli spinor fields with 3-dimensional rotations and vector fields. As an application, the problem of a classical particle in a central field of force is described in terms of one 2-component spinor field: the spinor equation of motion resembles the wave equation of quantum mechanics.

Key words: representations of vector spaces, rotations, Euler’s angles, 2-component spinor fields, quantum mechanics, wave equation, MAPLE.

1. Introduction

Quantum mechanics has been universally recognized as one of the chief achievements of physics of all times, and the physical predictions it offers are verified with a high degree of accuracy. However, as it is well known, it is not easy to understand: a number of difficulties even logical is implicated so it may appear more as a collection of rules useful to get correct physical predictions than a fully-developed theory. In the state of things it can be hoped that in spite of the peculiar features of quantum mechanics comparisons and formal analogies among the various physical theories help in making the quantum theory more clear. Notice that a better understanding would not only give an intellectual satisfaction but it would also provide a more sound basis to future developments of the theory.

The starting motivation of the present research is a rather modest one, but in the way of thinking outlined above. As it is known, spinor fields are usually employed in quantum theories; on the contrary, they are not employed in macroscopic physics where tensor (in particular, scalar and vector) fields are employed. Thus, the following questions naturally arise. What is the reason of a different formalism? Is any physics hidden in the very formalism? The affirmation by Penrose (1983) comes to mind: “we have still not yet seen the full significance of spinors—particularly the 2-components ones—in the basic structure of physical laws”.

To answer, understanding of both the tensor and spinor techniques is, obviously, needed, and it is well known that understanding of spinors has been a problem for many years. A partial reason for this is undoubtedly that many calculations with spinors are very cumbersome. Thus, in many cases they have not been fully developed and the results have been more guessed than exactly derived.

But now: (i) Pauli and Dirac spinor fields (i.e., the non-relativistic and relativistic
2-component spinor fields, respectively) and the related calculus have been clearly geometrically interpreted (cf. Piazzese (1992), (1993 a)), and (ii) using computer algebra is a common practice—which is both a relief and a safety in performing heavy calculations, although not necessarily a source of new great ideas...—. Thus, it appears that the time is ripe to start investigating the function of spinors in quantum physics.

This paper deals, first of all, with the (rather trivial) preliminary problem of describing (or “representing”) a vector space with an isomorphic space, preserving the consistency of the representation when either of spaces undergoes a linear transformation, and an exact definition of “representation” is introduced (cf. Section 2). (Strangely enough, a number of Authors dealing with spinors—in particular with the “ideal” approach—do not seem concerned about this point, which has been criticized by Piazzese (1993 b)). The representations of the real affine 3-dimension space (the usual “ambient” space of classical mechanics) with both space $\mathbb{R}^3$ and the real space of the traceless Hermitian matrices of second order are discussed, in Section 3. With the aid of such representations, descriptions are given of both proper rotations and vector fields in terms of Pauli spinor fields and the results are offered as explicit expressions of the Euler angles (cf. Sections 4 and 5). Finally, in Section 6, such spinor fields are shown at work, in an alternative description of the classical problem of a particle in a central field of force. Apparently, this is more than visualizing the spinors in some way, which has been the aim of some Authors even recently (cf. Ablamowicz, Lounesto, Maks (1990)).

All calculations have been performed with the aid of MAPLE. The results are clear, and, out of the mystique usually joined with spinors, they look trivial.

2. Representations of vector spaces

In dealing with linear transformation groups on vector spaces, two points of view, called the “active” and the “passive” points of view, are standard (cf. Goldstein (1980, page 137)). In the former, only the components are transformed, but not the basis. In the latter, the basis undergoes a transformation inverse of the one on the components. As a result, the active point of view describes an automorphism of the space. On the contrary, in the passive one the space elements are not changed. This Section deals with the problem of “describing” a vector space $V$ with another vector space $U$, in such a way that the description is preserved when either of spaces undergoes a linear transformation.

When only the passive point of view is taken into account, the foregoing requirement is clearly fulfilled if spaces $V$ and $U$ are isomorphic: a one-to-one correspondence $\omega$ exists, such that $U = \omega(V)$. But if we take into account also the active point of view, apparently the simplest possibility preserving the intrinsic feature of the description is requiring that, when an element $v \in V$ is transformed into another one $v' \in V$, a corresponding transformation $u \rightarrow u'$ occurs in $U$, in such a way that the one-to-one correspondence $\omega$ is preserved. In other words, we require