CHAPTER IV
SOME ONE-DIMENSIONAL DYNAMICAL PROBLEMS

We now consider some dynamical problems involving especially simple geometries: torsional oscillations of a rod, and propagation of plane waves in a semi-infinite medium. We have several purposes in mind. First, these problems illustrate some important similarities between the behavior of viscoelastic materials and the behaviors of materials that are purely viscous or purely elastic, and they also illustrate some important distinctions. Second, we wish to illustrate that viscoelasticity problems can be analyzed to significant depth even if the basic response functions are given only graphically or numerically, in the form of data. Third, the mathematical techniques used in these problems are of interest in themselves because they find applications in many areas.

Torsional oscillations are considered rather briefly. The greater part of this chapter is spent on the problem of one-dimensional pulse propagation and the various mathematical methods that can be illustrated in this context.

1. Torsional Oscillations.

We consider a rod of circular cross-section with radius $R$ and length $L$. One end of the rod is attached to a rigid support. A fly-wheel with moment of inertia $I$ is attached to the other end. At time zero the fly-wheel is turned through an angle $\theta_0$ and then released. We expect that its angular displacement $\theta(t)$ will then be a damped sinusoidal oscillation. Given the modulus $G(t)$ of the material of the rod, we wish to determine the frequency and damping coefficient for the oscillation. Conversely, we might wish to determine the material properties by observing the oscillation.

We neglect the inertia of the rod itself and ignore gravity. With neglect of inertia we can suppose that the rod is twisted uniformly through the angle $\theta(t)/L$ per unit length at any given time. (Would you be willing to believe this if the rod were very short? A mile long? What sort of phenomenon is being neglected?)
Each material element (bounded by cylindrical coordinate surfaces) is sheared in the azimuthal direction when the rod is twisted. At a distance \( r \) from the axis, the amount of shear is \( \kappa(r,t) = r\theta(t)/L \). (Show why, geometrically.) The shearing stress on a cross-section is then also azimuthal, with a resultant couple \( M(t) \), say, which is related to \( \theta(t) \) by

\[
M(t) = (\pi R^2/2L) \int_{-\infty}^{t} G(t-t')d\theta(t').
\]

(Problem: Derive this relation.)

The moment exerted on the fly-wheel by the rod is \(-M(t)\), so the equation of motion of the fly-wheel is

\[
I\ddot{\theta}(t) = -M(t).
\]

As initial conditions we suppose that \( \theta \) is zero for all negative times, and that at \( t = 0^+ \), \( \theta = \theta_0 \) and \( \dot{\theta} = 0 \). Then, for \( t > 0 \), we have

\[
\theta''(t) + F\theta_0 G(t) + F \int_{0}^{t} G(t-t')\theta'(t')dt' = 0,
\]

where \( F = \pi R^2/2LI \).

By applying the Laplace transform and rearranging, we obtain

\[
\frac{\Theta(s)}{\theta_0} = \frac{s}{s^2 + sFG(s)}.
\]

Our problem is to invert this transform. Before considering the general case, let's look at some examples.

a. Viscous fluid.

For purely viscous response there should be no oscillation at all. With \( G(t) = \eta_0 \delta(t) \), then \( \Theta(s) = \eta_0 \), and inverting the transform yields \( \theta(t)/\theta_0 = \exp(-(F\eta_0)t) \). This is obviously wrong. What is wrong with the derivation of it?