Before continuing with our study of the elementary properties of vector spaces and their linear subspaces let us collect a list of examples of vector spaces. We have already encountered the cartesian $k$-space $\mathbb{R}^k$ and so for the sake of completeness let us begin by listing this example:

**Example 1. $\mathbb{R}^k$**

The first new example that we have in this chapter is primarily designed to destroy the belief that a vector is a quantity with both direction and magnitude and to give meaning to the phrase in our comments on axiomatics in Chapter 2, that “the possibility is opened of assigning to them (the axioms of a vector space) content in new and unforeseen ways.”

**Example 2. $\mathcal{P}_n(\mathbb{R})$**

The vectors in $\mathcal{P}_n(\mathbb{R})$ are polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$

of degree less than or equal to $n$, that is, $m \leq n$. Addition of vectors is to be ordinary addition of polynomials and multiplication of a polynomial by a number, the ordinary product of a polynomial by a number. With these interpretations of the basic terms:

- vector $\leftrightarrow$ polynomial of degree $\leq n$
- vector addition $\leftrightarrow$ addition of polynomials
- scalar multiplication $\leftrightarrow$ multiplication of a polynomial by a number,

we obtain an example of a vector space. To verify that $\mathcal{P}_n(\mathbb{R})$ is indeed a vector space we must check that the eight declarative sentences obtained
from these interpretations of the basic terms vector, vector addition, scalar multiplication, are true sentences. This is a straightforward deduction from the (assumed) properties of real numbers following the pattern of (2.4) and will be left to the diligent reader.

Note that in this example it is very difficult to say what direction or length a vector has.

**Example 3. C**

Let us denote by C the *complex numbers*. (We will not here be concerned with the technical details of constructing the complex numbers, but will take them as we learned them in grammar school.) Recall that a complex number looks like

\[ a + bi \]

where \( a, b \) are real numbers, and \( i \) is a number with \( i^2 = -1 \).

The vectors in our vector space will be complex numbers. Addition of vectors is to be the ordinary addition of complex numbers, and scalar multiplication the familiar process of multiplying a complex number by a real number. With these interpretations of the basic terms

- vector \( \leftrightarrow \) complex number
- vector addition \( \leftrightarrow \) addition of complex numbers
- scalar multiplication \( \leftrightarrow \) multiplication of a complex number by a real number,

we obtain an example of a vector space. The verifications are again routine.

Note that in Example 3 we are not using all of the structure that we have, for it is possible to multiply two complex numbers, that is, *in this example* we may multiply two vectors, something it is not always possible to do in a vector space. This is a possibility worthy of further study, and we will do just that when we study spaces of linear transformation and *linear algebras*.

Note. the product of two polynomials of degree at most \( n \) will have degree at most \( 2n \), so you cannot multiply elements of \( P_n(\mathbb{R}) \) in any obvious way.

Example 2 is a very important example, and a prototype for many others of the same type. These examples are characterized by the fact that their "vectors" are actually functions of some type or other.

**Example 4. \( P(\mathbb{R}) \)**

The simplest way to obtain a space akin to but different from \( P_n(\mathbb{R}) \) is simply to remove the restriction that the polynomials have degree at most \( n \). In this way we obtain the vector space \( P(\mathbb{R}) \) whose vectors are the polynomials

\[ p(x) = a_0 + a_1x + \cdots + a_m x^m \]