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The Laplace equation*

1. Green’s Identity, Fundamental Solutions, and Poisson’s Equation†

The Laplace operator acting on a function $u(x) = u(x_1, \ldots, x_n)$ of class $C^2$ in a region $\Omega$ is defined by

$$\Delta = \sum_{k=1}^{n} D_k^2$$

(1.1)

For $u, v \in C^2(\overline{\Omega})$ we have‡ (see Chapter 3, (4.8), (4.9)) Green’s identities.

$$\int_{\Omega} v \Delta u \, dx = - \int_{\Omega} \sum_{i} v_{x_i} u_{x_i} \, dx + \int_{\partial\Omega} v \frac{du}{dn} \, dS$$

(1.2a)

$$\int_{\Omega} v \Delta u \, dx = \int_{\Omega} u \Delta v \, dx + \int_{\partial\Omega} \left( v \frac{du}{dn} - u \frac{dv}{dn} \right) \, dS,$$

(1.2b)

where $d/dn$ indicates differentiation in the direction of the exterior normal to $\partial\Omega$.

The special case $v = 1$ yields the identity

$$\int_{\Omega} \Delta u \, dx = \int_{\partial\Omega} \frac{du}{dn} \, dS.$$  

(1.3)

*([2], [6], [11], [13], [14], [14], [23], [26])

†([17])

‡We assume here that $\Omega$ is an open bounded set, for which the divergence theorem, (4.1) of Chapter 3, is valid.

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Another special case of interest is \( v = u \). We find then from (1.2a) the

**energy identity**

\[
\int_\Omega \sum_i u_i^2 \, dx + \int_\Omega u \Delta u \, dx = \int_{\partial \Omega} u \frac{du}{dn} \, dS. \tag{1.4}
\]

If here \( \Delta u = 0 \) in \( \Omega \) and either \( u = 0 \) or \( du/dn = 0 \) on \( \partial \Omega \), it follows that

\[
\int_\Omega \sum_i u_i^2 \, dx = 0. \tag{1.5}
\]

For \( u \in C^2(\Omega) \) the integrand is nonnegative and continuous, and hence has to vanish. Thus \( u = \text{const.} \) in \( \Omega \). This observation leads to uniqueness theorems for two of the standard problems of potential theory:

The **Dirichlet problem**: Find \( u \) in \( \Omega \) from prescribed values of \( \Delta u \) in \( \Omega \) and of \( u \) on \( \partial \Omega \).

The **Neumann problem**: Find \( u \) in \( \Omega \) from prescribed values of \( \Delta u \) in \( \Omega \) and of \( du/dn \) on \( \partial \Omega \).

As always in discussing uniqueness of linear problems, we form the difference of two solutions, which is a solution of the same problem with data 0. We find that the difference is a constant, which, in the Dirichlet case, must have the value 0. Thus: A solution \( u \in C^2(\Omega) \) of the Dirichlet problem is determined uniquely. A solution \( u \in C^2(\Omega) \) of the Neumann problem is determined uniquely within an additive constant. (Notice also that the solution of the Neumann problem can only exist if the data satisfy condition (1.3)).

One of the principal features of the Laplace equation

\[
\Delta u = 0 \tag{1.6}
\]

is its spherical symmetry. The equation is preserved under rotations about a point \( \xi \), that is under orthogonal linear substitutions for \( x - \xi \). This makes it plausible that there exist special solutions \( v(x) \) of (1.6) that are invariant under rotations about \( \xi \), that is have the same value at all points \( x \) at the same distance from \( \xi \). Such solutions would be of the form

\[
v = \psi(r), \tag{1.7}
\]

where

\[
r = |x - \xi| = \sqrt{\sum_i (x_i - \xi_i)^2} \tag{1.8}
\]

represents the euclidean distance between \( x \) and \( \xi \). By the chain rule of differentiation we find from (1.6) in \( n \) dimensions that \( \psi \) satisfies the ordinary differential equation

\[
\Delta \psi = \psi''(r) + \frac{n-1}{r} \psi'(r) = 0. \tag{1.9}
\]