CHAPTER 14. INFERENCES FOR NORMAL DISTRIBUTION PARAMETERS

The normal distribution has many important applications in Statistics. Many types of measurements appear as though they could have come from normal distributions. The Central Limit Theorem (Section 6.7) justifies the assumption of a normal distribution, at least as a first approximation, in many situations.

Another reason for the widespread use of normal distribution models is that the corresponding statistical procedures are relatively simple, and multiparameter problems can be dealt with more easily than for most other distributions. Because of this, there is a tendency to use normal theory whenever possible. Even if the original measurements are decidedly non-normal, it may be possible to find a nonlinear transformation such that normal distribution results can be applied to the transformed measurements. An example of this is the analysis of log-lifetimes as normally distributed measurements in many engineering applications. Quantile plots (Section 11.5) may be useful in checking the assumption of normality for the original or transformed measurements.

Unfortunately, it is not uncommon for normal theory to be applied in situations where the assumptions clearly are not met. For instance, it is not difficult to find examples in the scientific literature where 0-1 variables have been analysed as though they were continuous and normally distributed! There might have been an excuse for this twenty years ago because of computational problems with alternate models. Modern computers have changed this, and many alternate methods of analysis are now available for cases in which normality assumptions would be inappropriate.

In Section 1 we describe some general methods to be used in the remainder of this chapter and in Chapter 15. The analysis of a sample of measurements from a single normal distribution with unknown mean and variance is considered in Section 2, and Section 3 discusses the analysis of paired data (e.g. before and after measurements). Sections 4 and 5 deal with the comparison of means and variances of two independent normal samples, and Section 6 considers these problems for k normal samples.

Tests and confidence intervals for normal means are obtained from standardized normal and t distributions. Inferences for normal variances involve \( \chi^2 \) and F distributions. The reader might wish to
go over the material in Section 6.9 again before continuing with this chapter.

14.1 Introduction

In this chapter and the next one we shall consider several situations involving the analysis of measurements which are assumed to be independent and normally distributed. Usually the measurements are assumed to have the same unknown variance $\sigma^2$. The analysis is similar if the variances are known multiples of a single unknown variance $\sigma^2$.

The parameter $\phi$ which is of primary interest is usually related to the means of the assumed normal distributions. For instance, $\phi$ might equal a single unknown mean, a difference of two unknown means, or more generally, a linear function of several unknown means.

Under the assumptions described above, the MLE $\hat{\phi}$ will be normally distributed, $\hat{\phi} \sim N(\phi, \sigma^2)$, where $c$ is a known constant, so that

$$Z \equiv \frac{\hat{\phi} - \phi}{\sigma \sqrt{c^2}} \sim N(0,1). \quad (14.1.1)$$

Furthermore, there will exist a statistic $V$, say, which carries all of the information about $\sigma^2$, where $V$ and $\hat{\phi}$ are distributed independently of one another, and

$$U \equiv \frac{V}{\sigma^2} \sim \chi^2_v. \quad (14.1.2)$$

The degrees of freedom $v$ will depend upon the particular situation. We shall show that the MLE of $\sigma^2$ based on the distribution of $V$ is

$$S^2 \equiv \frac{V}{v}. \quad (14.1.3)$$

If $\sigma$ is known, tests and confidence intervals for $\phi$ would be based on (14.1.1). Either a two-tail test or a likelihood ratio test for $H: \phi = \phi_0$ would give

$$SL(\phi_0) = P\{|Z| \geq |z_{obs}|\} \quad \text{where} \quad z_{obs} = \frac{\hat{\phi} - \phi_0}{\sqrt{\sigma^2}}. \quad (14.1.4)$$

Since $P\{|Z| \geq 1.960\} = 0.05$ from Table B1, it follows that $SL \geq 0.05$ if and only if $|z_{obs}| \leq 1.960$. Hence the 95% confidence interval for $\phi$ is given by

$$-1.960 \leq \frac{\hat{\phi} - \phi}{\sigma \sqrt{c^2}} \leq 1.960 \iff \phi \in \hat{\phi} \pm 1.960\sqrt{c^2}. \quad (14.1.4)$$